Acquiring Maps of Interrelated Conjectures on Sharp Bounds

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ABSTRACT

To automate the discovery of conjectures on combinatorial objects, we introduce the concept of a map of sharp bounds on characteristics of combinatorial objects, that provides a set of interrelated sharp bounds for these combinatorial objects. We then describe a Bound Seeker, a CP-based system, that gradually acquires maps of conjectures. The system was tested for searching conjectures on bounds on characteristics of digraphs: it constructs sixteen maps involving 431 conjectures on sharp lower and upper-bounds on eight digraph characteristics.

PART I: CONTEXT AND QUESTIONS

COMBINATORIAL OBJECTS AND THEIR CHARACTERISTICS

**DIGRAPHS**
- \( a = 16 \)
- \( v = 4 \)
- \( c = 8 \)
- \( \) \( a = \) number of arcs
- \( v = \) number of vertices
- \( c = \) number of connected components

**FORESTS**
- \( f = 3 \)
- \( t = 1 \)
- \( \) \( f = \) number of leaves
- \( t = \) number of trees

QUESTIONS

How to acquire from data representing instances of combinatorial objects:

1. Sharp bounds of characteristics?
2. Relations between sharp bounds?
3. And organise (1) and (2) into a map?

Illustrating the questions wrt digraphs

1. Sharp bounds:
   - \( a \leq (c - (c - 3))^2 + (c - 1) \)
   - \( \) \( f = 6 \)
   - \( t = 2 \)
   - \( v = 3 \)
   - \( z = 3 \)

2. Relation:
   - If \( a = v^2 \) then \( a = 1 \)

3. A map:

PART II: DEFINITION OF A MAP OF CONJECTURES

Given a finite set of input characteristics \( P \) and an output characteristic \( o \in P \), a map of sharp upper bounds \( \Delta P \) is defined as a digraph where:

- Each node of the map is associated with a subset \( P \subseteq P \) of input characteristics and corresponds to a maximum conjecture of the form \( o \leq f(P) \), where \( f \) is a function of \( P \).

This inequality is tight, i.e. there exist values that can be given to the parameters \( P \) in order to reach the equality.

- Each arc from conjecture \( o \leq f_i(P \cup \{q\}) \) to conjecture \( o \leq f_j(P) \) corresponds to a projection from a subset \( P \subseteq P \cup \{q\} \) of input characteristics to a subset \( P \) of input characteristics, by eliminating a characteristic \( q \).

The equality \( q = g(P) \) is called a maximality conjecture.

ILLUSTRATING THE DEFINITION OF A MAP

\[ o \leq f_1(P \cup \{q\}) \rightarrow q = g(P) \rightarrow o \leq f_2(P) \]

- Maximum conjecture 1
- Maximality conjecture
- Maximum conjecture 2

\[ (A) \quad \rightarrow \quad (B) \quad \rightarrow \]

\[ v = 3 \quad \Rightarrow \quad q = 3 \quad \Rightarrow \quad o = 9 \]

\[ f_1 = \text{upper bound of } q \text{ wrt } P \cup \{q\} \]

\[ f_2 = \text{upper bound of } o \text{ wrt } P \]

\[ q = \text{linking characteristic} \]

\[ g = \text{a function of } P \]

PART III: RESULTS

Comparison between the Bound Seeker (BS) and the bounds of the Global constraints catalog (GCC)

<table>
<thead>
<tr>
<th>Number of bounding characteristics</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent sharp bounds retrieved by BS</td>
<td>22</td>
<td>14</td>
<td>4</td>
<td>40</td>
</tr>
<tr>
<td>Sharper bounds than the GCC found by BS</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Generalised sharp bounds found by BS</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>Erroneous bounds found in the GCC by BS</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Bounds in the GCC not retrieved by BS</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Total bounds of the GCC for each column</td>
<td>24</td>
<td>24</td>
<td>12</td>
<td>60</td>
</tr>
</tbody>
</table>

Map of 16 sets of bounding characteristics used to bound the characteristic \( \zeta \)

- \( \zeta \leq v \quad \Rightarrow \quad v = 3 \quad \Rightarrow \quad o = 9 \)
  - \( \zeta \leq v - c + 1 \)
  - \( \zeta \leq v - c \quad \Leftrightarrow \quad v - c \)
  - \( \zeta \leq v - c \quad \Rightarrow \quad v - c \)
  - \( \zeta = v - c + 1 \quad \Rightarrow \quad v - c \quad \Rightarrow \quad o = 9 \)
  - \( \zeta = v + c \quad \Rightarrow \quad v + c \quad \Rightarrow \quad o = 9 \)
  - \( \zeta = v + c \quad \Rightarrow \quad v + c \quad \Rightarrow \quad o = 9 \)
  - \( \zeta = v + c \quad \Rightarrow \quad v + c \quad \Rightarrow \quad o = 9 \)
  - \( \zeta = v + c \quad \Rightarrow \quad v + c \quad \Rightarrow \quad o = 9 \)

- \( v = \) number of vertices
- \( c \) : number of connected components
- \( s \) : number of strongly connected components
- \( \zeta \) : size of the smallest connected component
- \( \delta \) : size of the largest strongly connected component
- \( g \) : size of the smallest strongly connected component

- \( \zeta \leq v - c + 1 \)
- \( \zeta \leq v - c \quad \Rightarrow \quad v - c \)
- \( \zeta \leq v - c \quad \Rightarrow \quad v - c \)
- \( \zeta = v - c + 1 \quad \Rightarrow \quad v - c \quad \Rightarrow \quad o = 9 \)
- \( \zeta = v + c \quad \Rightarrow \quad v + c \quad \Rightarrow \quad o = 9 \)
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