The Multi-Inter-Distance Constraint

Pierre Ouellet and Claude-Guy Quimper

Introduction

- The MULTI-INTER-DISTANCE constraint is a new global constraint.
- It is useful to model scheduling problems.
- We present a filtering algorithm achieving bounds consistency.
- The filtering algorithm relies on the theory of the shortest paths in a graph.
- We experimented on the runway scheduling problem.

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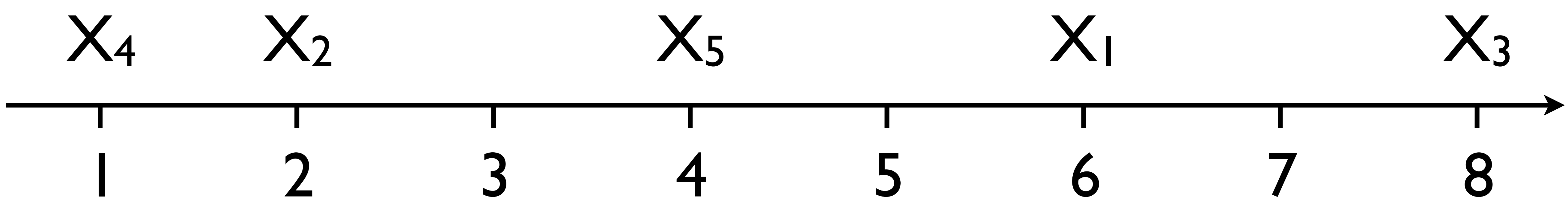

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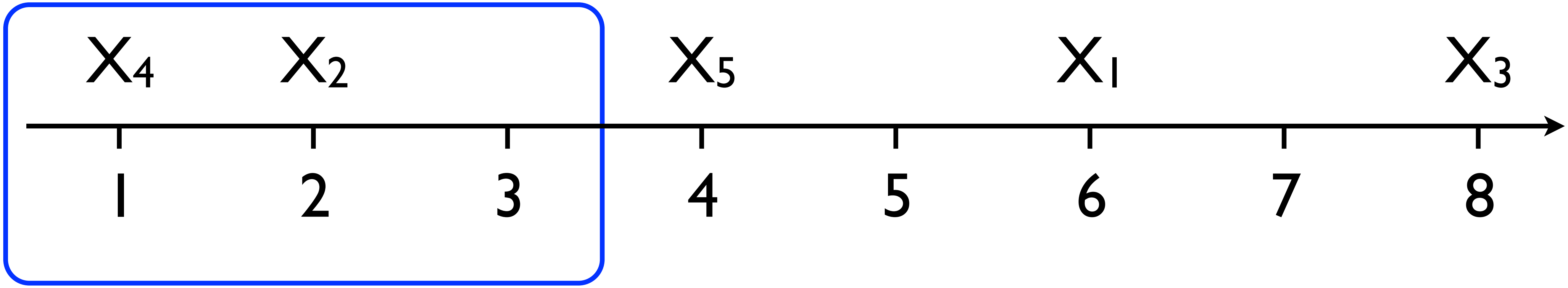

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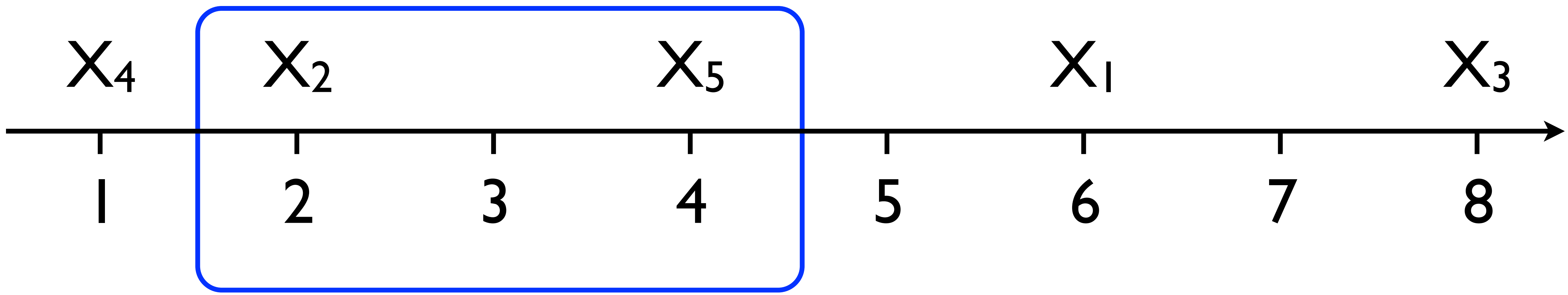

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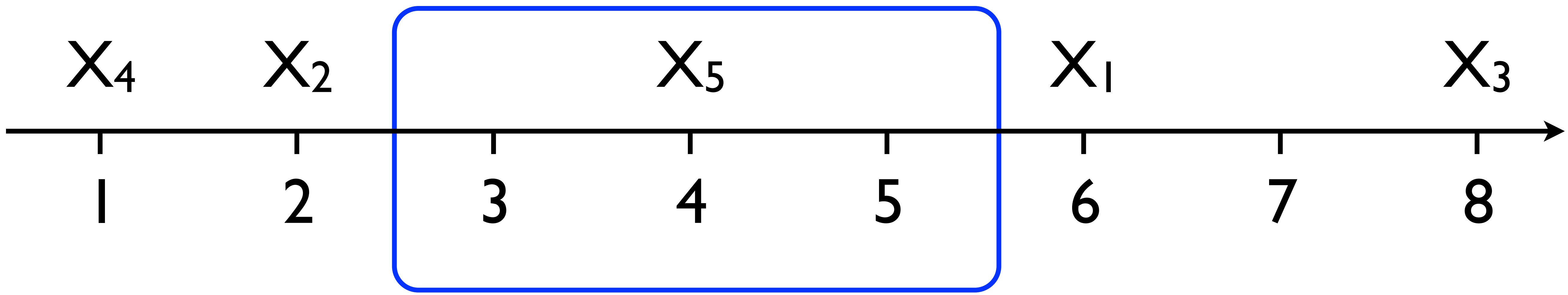

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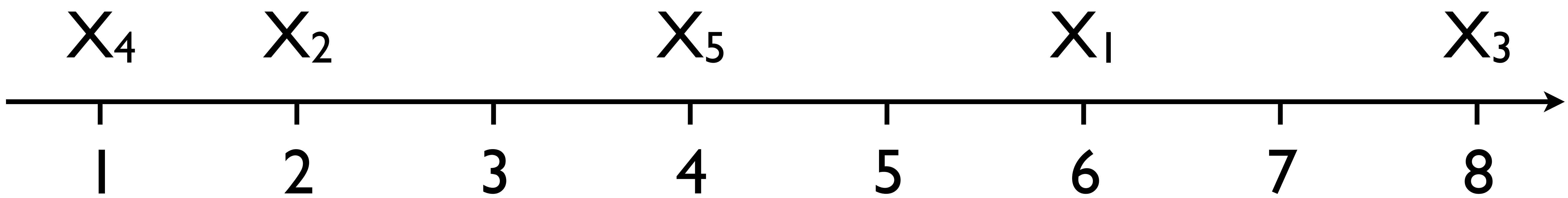

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  - For $m = 1$ and $p = 1$, the MULTI-INTER-DISTANCE constraint encodes the ALL-DIFFERENT constraint.
  - When $m = 1$, the constraint specializes into the INTER-DISTANCE constraint.

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- **Domain consistency** is NP-Hard to enforce as it is for the Inter-Distance constraint. [Artiouchine and Baptiste 2005]
- We show how to enforce **bounds consistency** in polynomial time.
  - We assume that the domain of a variable X_i is an interval $[l_i, u_i)$.
  - We want to shrink this interval to remove all values that are not involved in any solution.

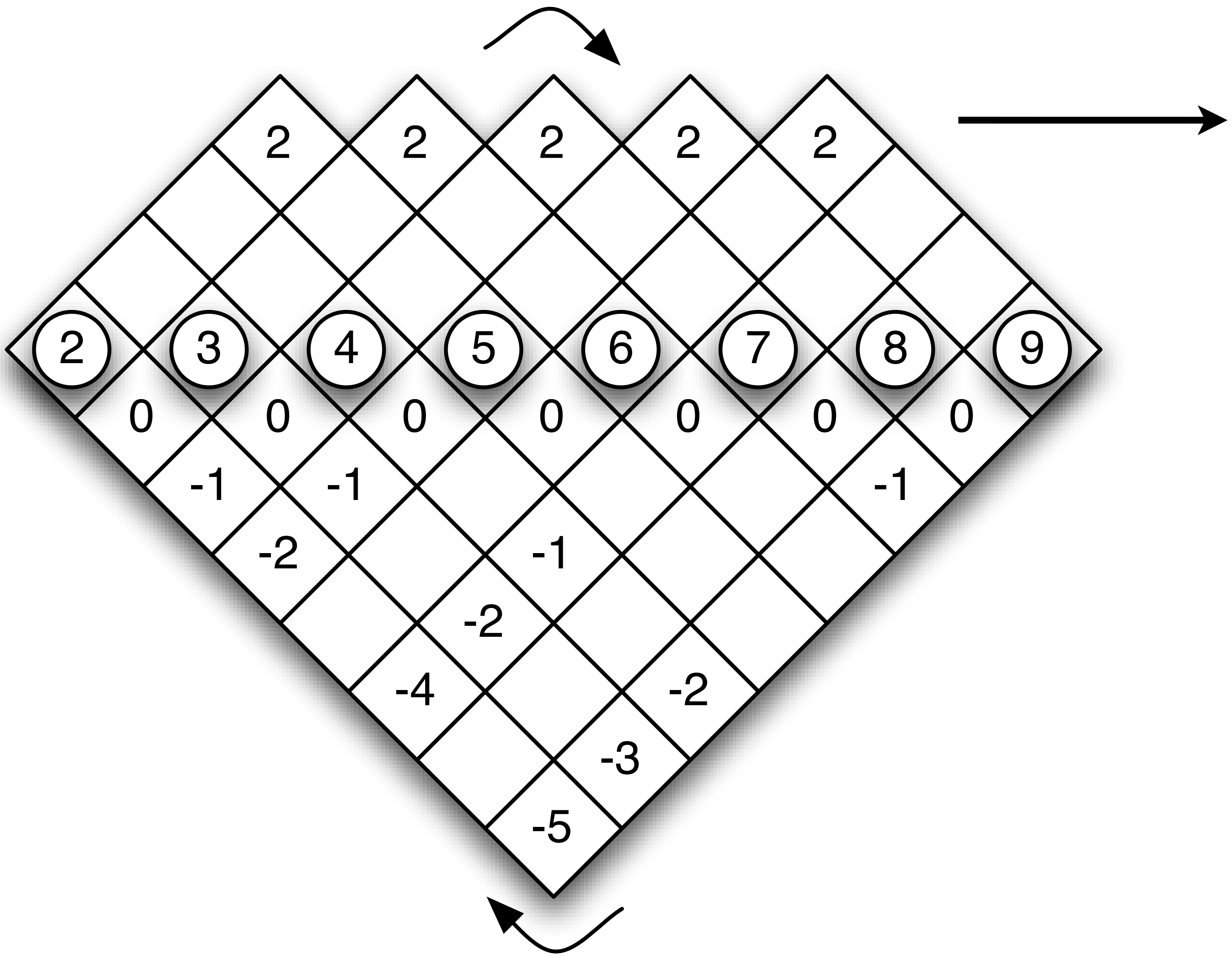
Test for Satisfiability

- The Multi-Inter-Distance constraint is satisfiable iff the following scheduling problem has a solution:
  - Task *i* starts at or after time l_i but before time u_i
  - Task *i* is executed without preemption for *p* units of time
  - Task *i* does not overload one of the *m* resources.
- This scheduling problem is solved in time $O(n^2 \min(1, p/m))$ [López-Ortiz & Quimper, 2011].
- We use this scheduling algorithm as a sub-routine in our filtering algorithm.

Scheduling Graph

$$\textsc{Multi-Inter-Distance}([X_1, \ldots, X_5], m = 2, p = 3)$$

$$X_1 \in [7, 9),\ X_2 \in [2, 4),\ X_3 \in [4, 7),\ X_4 \in [2, 7),\ X_5 \in [3, 5)$$



Forward Edges

Connect two time points that are ***p*** units of time apart with an edge of weight ***m***.

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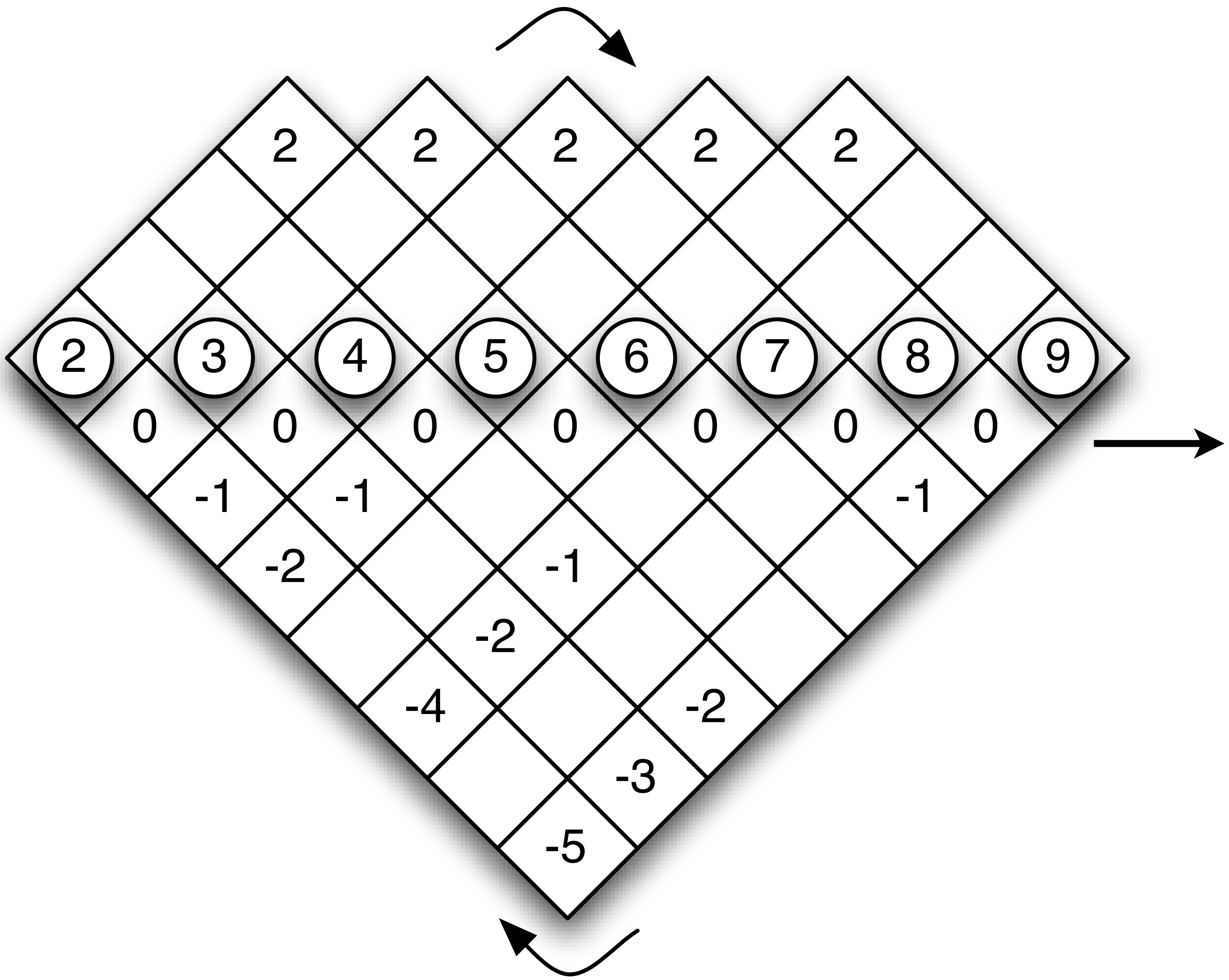

Null Edges

Connect a time point with its predecessor with an edge of weight **0**.

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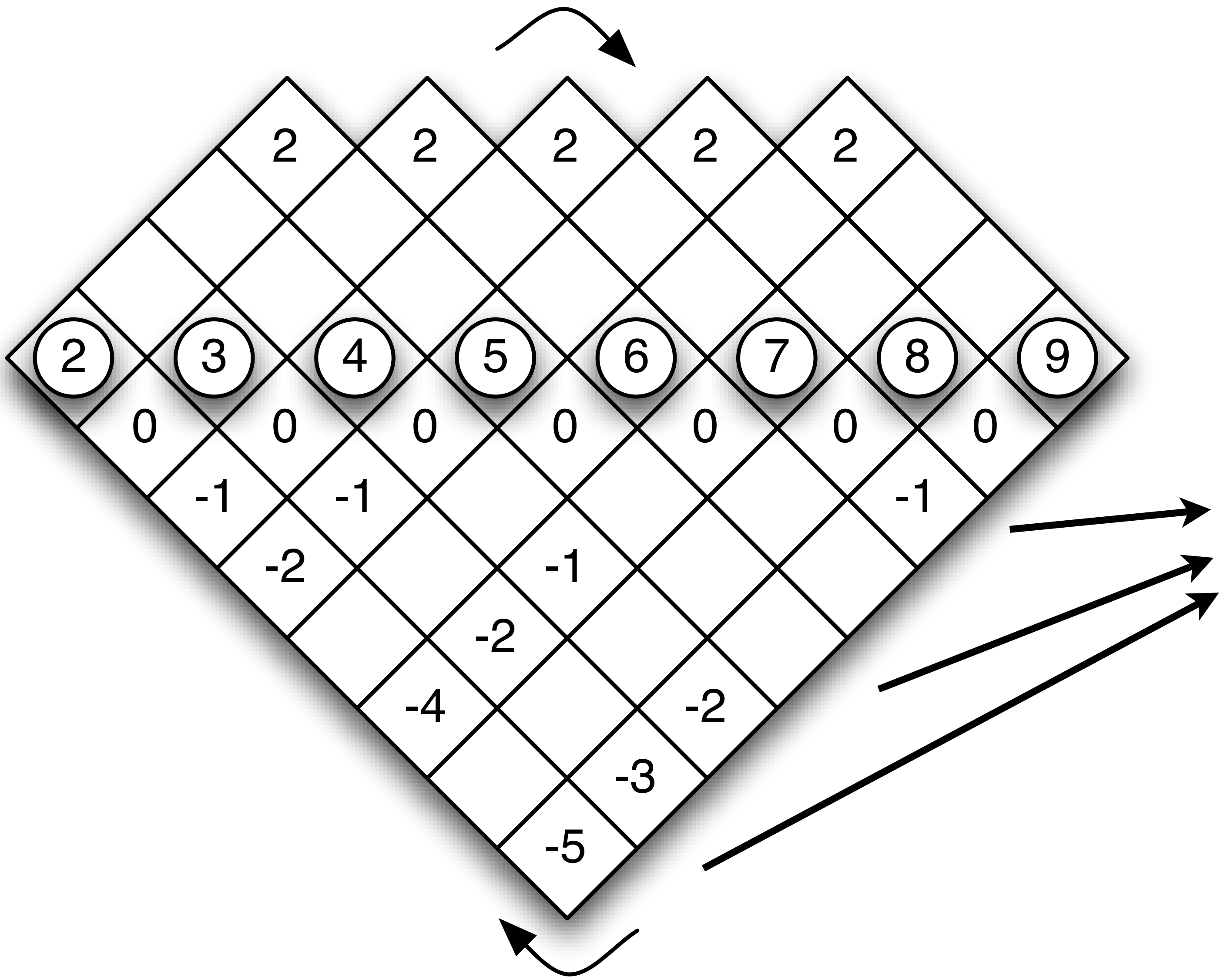

Backward Edges

Connect an upper bound with a lower bound. The absolute value of the weight is the number of domains contained in the interval spanned by the edge.

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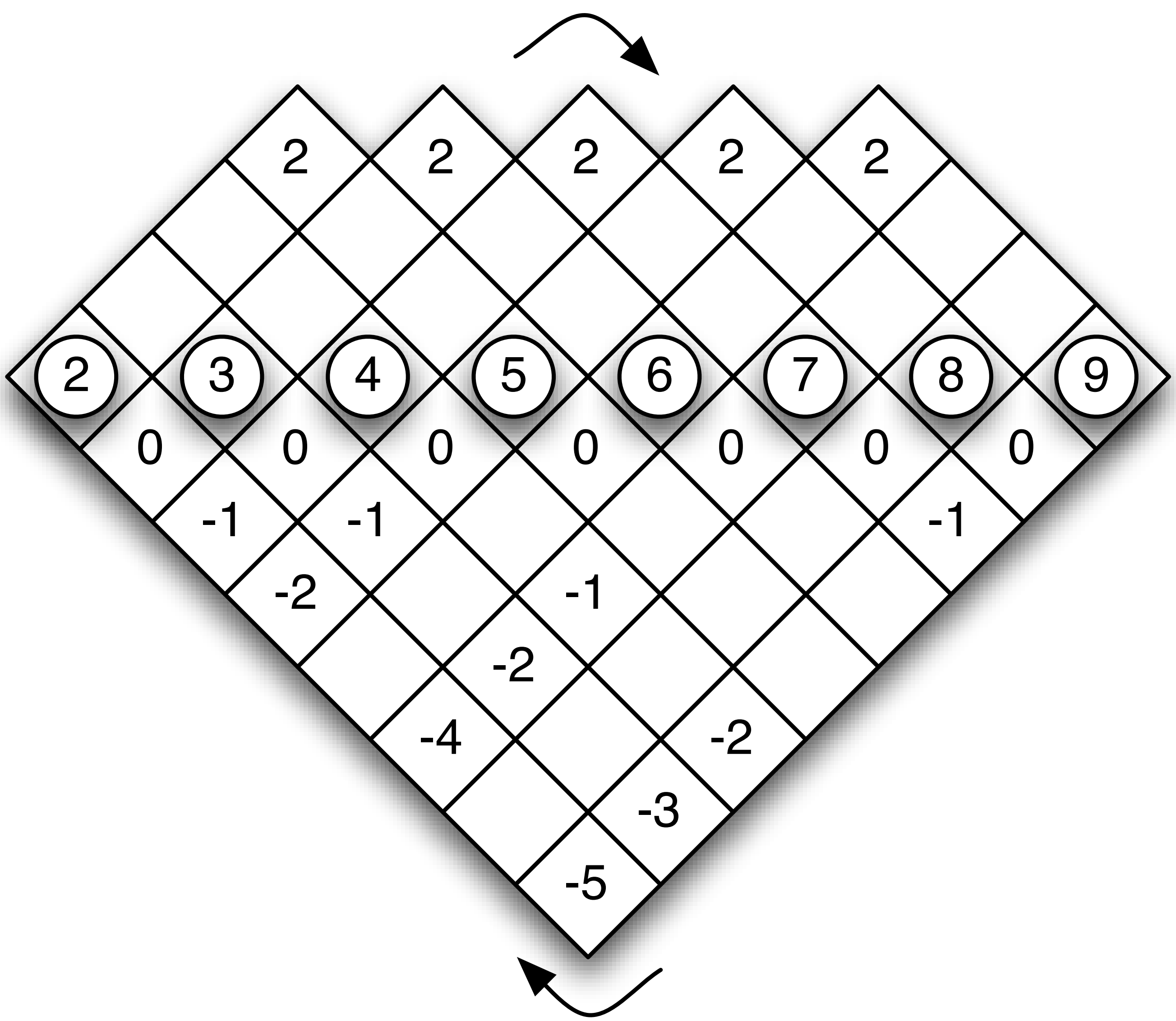
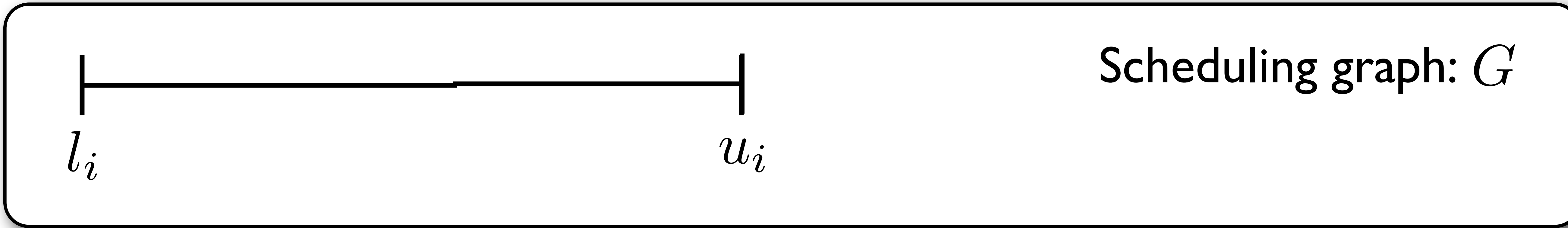


Theorem

The Multi-Inter-Distance constraint is satisfiable if and only if the scheduling graph has no negative cycles.

[Dürr & Hurand 2009]

First Pruning Rule



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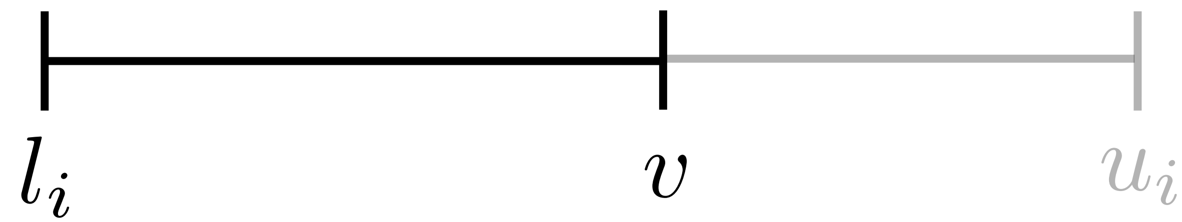

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Altered scheduling graph: G_i^v

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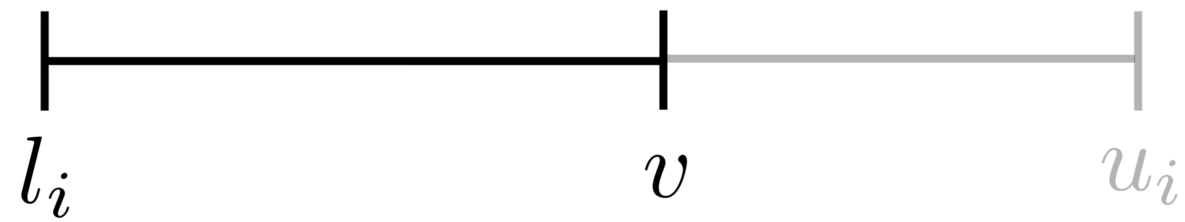

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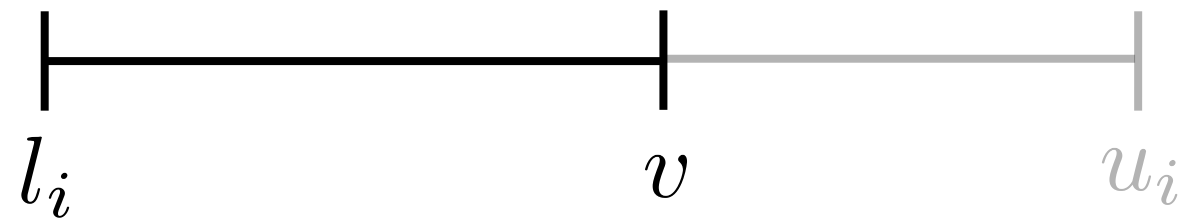

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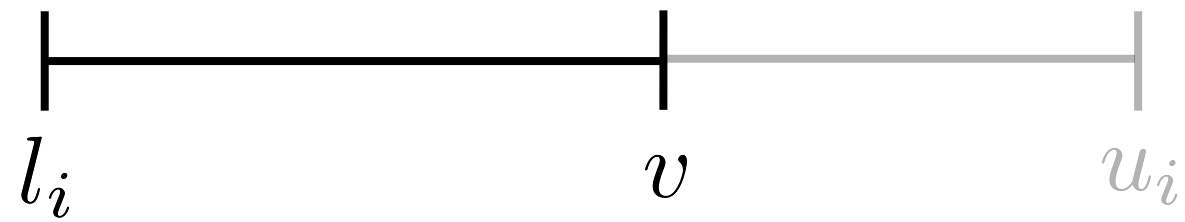

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- **Rule:** Lower bounds in that forbidden region should be increased to *v*.

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- **Theorem**: The smallest value that has a support in dom(X_i) is the largest value that is at distance 0 from l_i in $G_i^{u^*}$.
- **Rule:** Compute the shortest paths from l_i to all the other nodes. Set the new lower bound to the largest value that is at distance 0 from l_i.

Filtering Algorithm

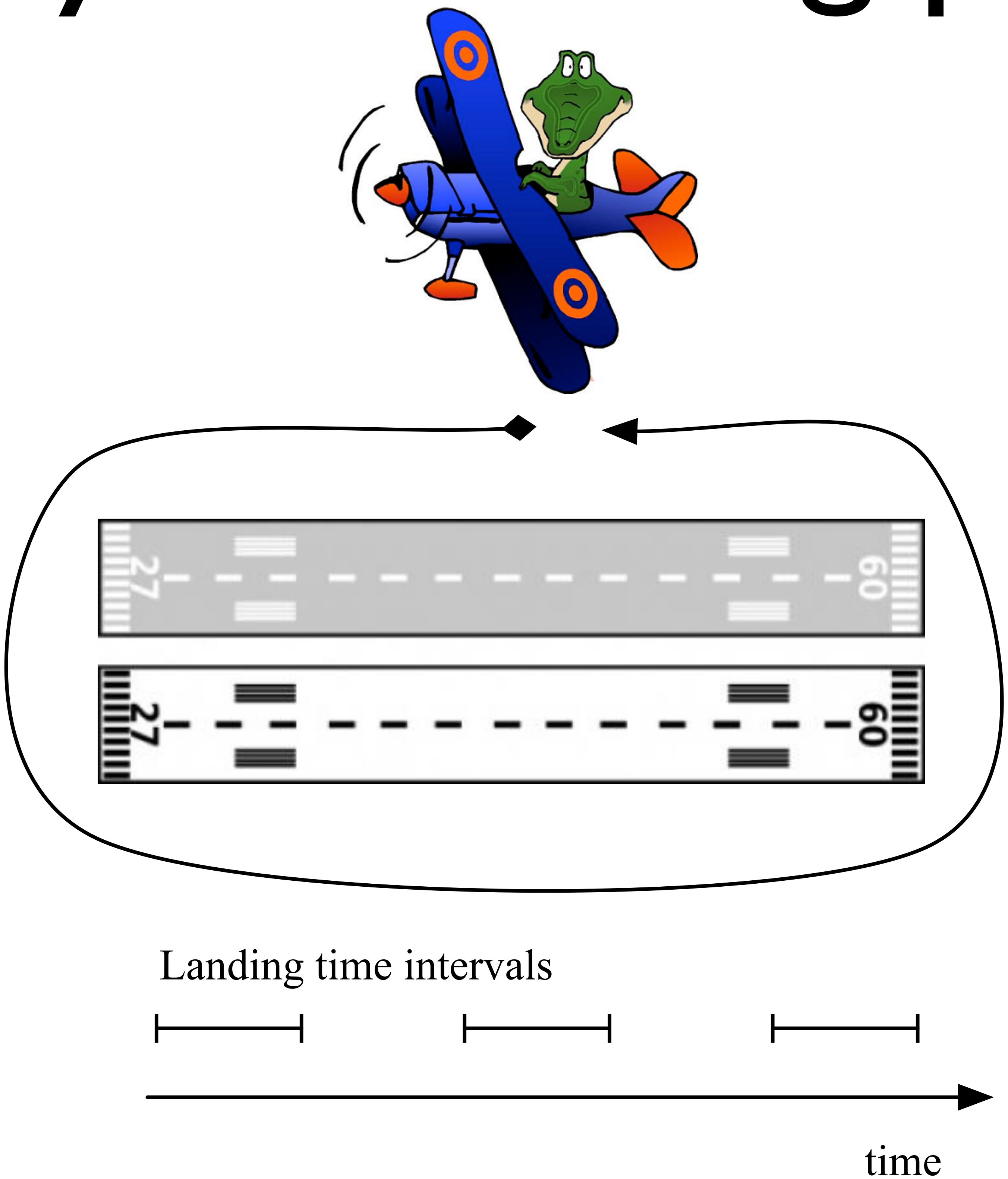
We process the variables in non-decreasing order of upper bounds.

1. Let the interval $[l_i, u_i)$ be the domain of the variable X_i.
2. Let u^* be the smallest domain upper bound greater than l_i.
3. If the altered scheduling graph $G_i^{u^*}$ has a negative cycle, the interval $[l_i, u^*)$ is a forbidden region and we prune the domains accordingly. Go to 2.
4. If the altered scheduling graph has no negative cycles, let *v* be the largest value at distance 0 from l_i.
5. Set the lower bound of X_i to v.
6. Process next variable.

Running Time Complexity

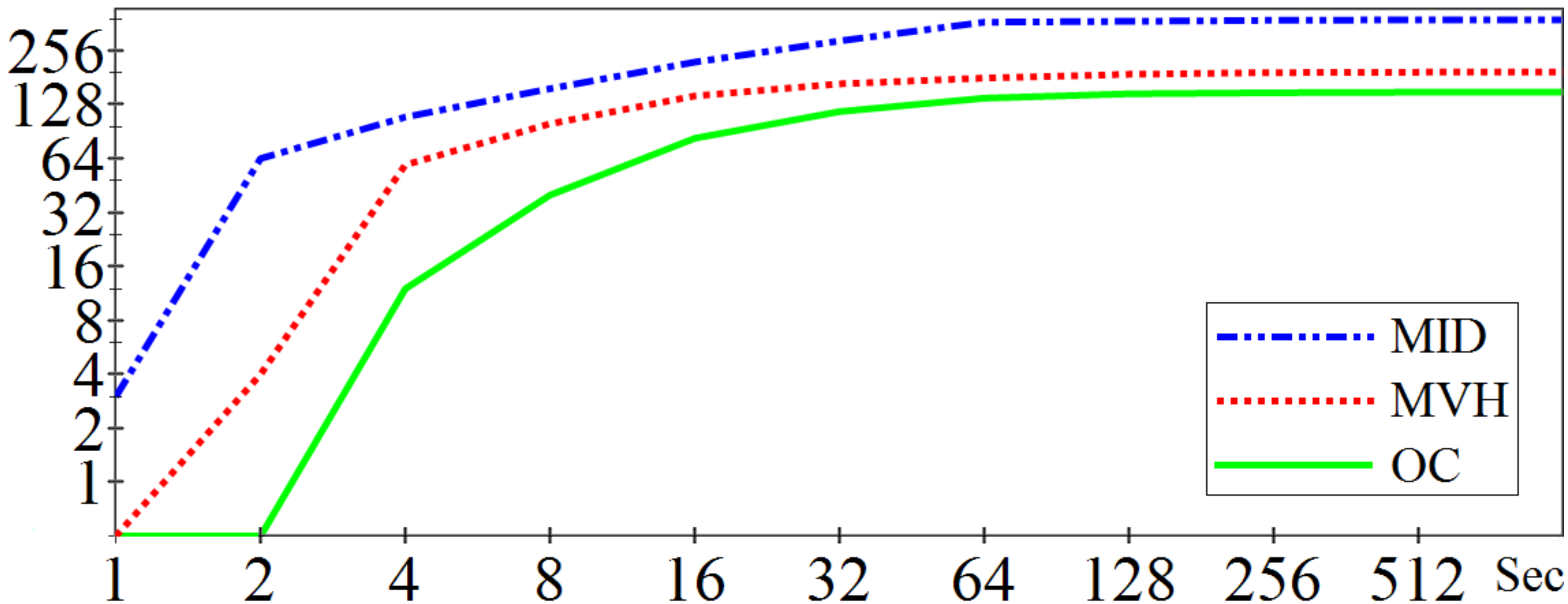
- Computing a shortest path: $O(n^2 \min(1, \frac{p}{m}))$
- Maximum number of shortest path computations: $2n$
- Total running time complexity: $O(n^3 \min(1, \frac{p}{m}))$

Runway scheduling problem



One runway

Number of instances solved vs time



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MVH	Edge-Finder [Mercier & Van Hentenryck]	$O(n^2)$
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Two or Three Runways

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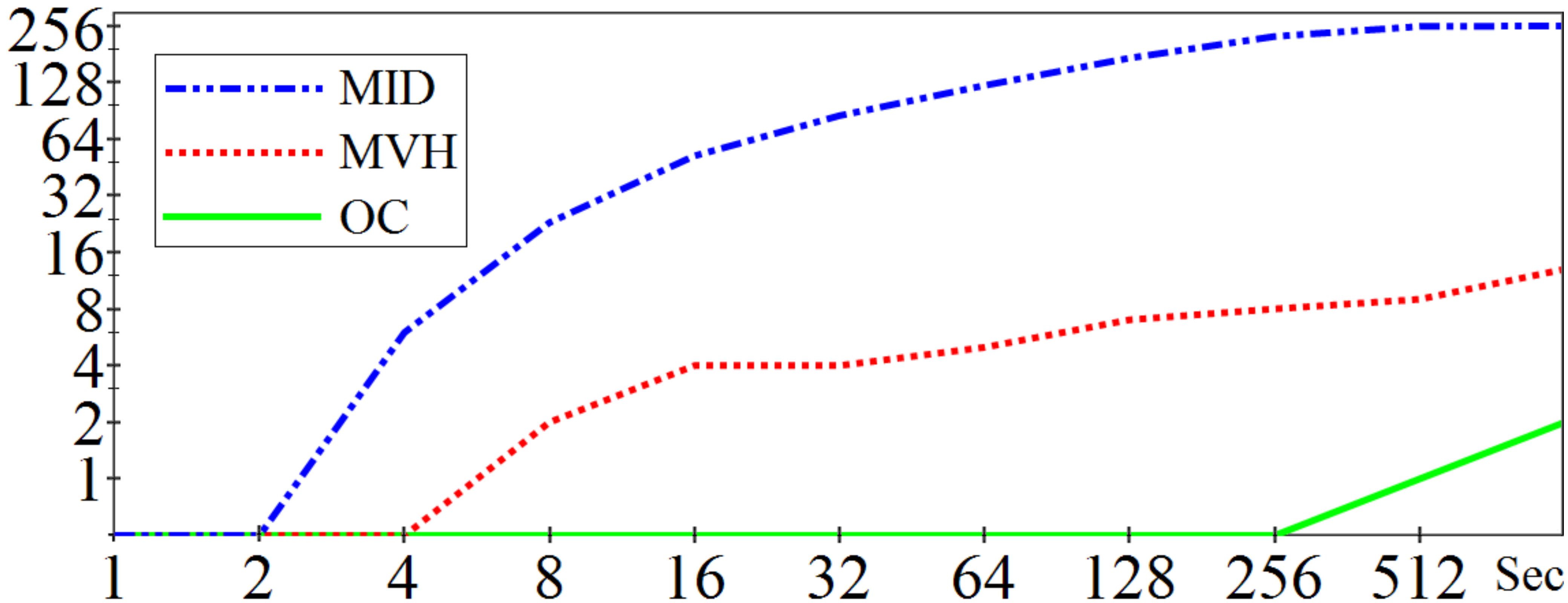

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Conclusion

- The Multi-Inter-Distance constraint is a new constraint that models certain scheduling problems.
- We showed how to enforce bounds consistency in polynomial time.
- The filtering algorithm relies on the properties of shortest paths in the scheduling graph.
- Experiments on the runway scheduling problem proved that a strong consistency is necessary to efficiently solve the problem.