Abstract

We show that a graph can always be decomposed into edge-disjoint subgraphs of countable cardinality in which the edge-connectivities and edge-separations of the original graph are preserved up to countable cardinal. We also show that this result, with the assumption of the Generalized Continuum Hypothesis, can be generalized to any uncountable cardinal. As applications of such decompositions we prove some results about Seymour’s double cover conjecture for infinite graphs, and about the maximal number of edge-disjoint spanning trees in graphs having high edge-connectivity. However, the main motivation for introducing these decompositions can be found in the second part of this paper where, to achieve a complete solution of the circuit decomposition problem (i.e.: the problem of characterizing the graphs that admit decompositions into 2-regular connected subgraphs), we use the results of this first part to carry out a reduction to the countable case.

1 Introduction

Many problems in infinite graph theory have quite simple solutions in the countable case whereas in the uncountable case the solution may be extremely complicated or the problem may even remain a conjecture. Such a problem is often solved by finding a way to decompose the whole graph into smaller fragments that preserve some specific properties of the original graph and are such that a solution of the problem for the fragments gives rise to a solution for the whole graph.

In this paper, we study decompositions of this kind. Our convention is that a decomposition is an equivalence relation on $E(G)$ such that every fragment (i.e., subgraph induced by the edges of an equivalence class) is connected. We are interested in finding decompositions whose fragments inherit as far as possible the
edge-connectivity of the original graph in the sense that for a given infinite cardinal \(|\alpha|\), the fragments of the decomposition are all of order at most \(|\alpha|\) and are such that no bond (i.e. cocycle) of cardinality \(|\leq \alpha|\) is split into pieces belonging to different fragments: such decompositions will be called bond-faithful \(|\alpha|\)-decompositions.

The main result of the paper (Theorem 3) is that for any graph \(G\) and any \(\alpha \geq \omega\), one can always construct a bond-faithful \(|\alpha|\)-decomposition.

We also introduce some applications of this theorem; one is that a graph \(G\) can always be split into two edge-disjoint parts \(K\) and \(L\) such that for each pair \(x, y\) of infinitely edge-connected vertices of \(G\), the edge-connectivity between \(x\) and \(y\) is the same in all three graphs \(G\), \(K\) and \(L\). Another application is that an \(\alpha\)-edge-connected graph always contains \(\alpha\) edge-disjoint spanning trees. Moreover, in the second part of this paper [3] that is specifically devoted to decompositions into (finite and infinite) circuits, we use Theorem 3 to carry out a reduction to the countable case which is considerably easier to handle. This application was the main motivation for introducing the concepts and proving the main result of the present paper.

Some preliminary results about Theorem 3 also have interesting consequences bearing on Seymour’s double-cover conjecture [6], saying that the conjecture is true for any graph (finite or infinite) provided it is true for the 3-regular ones. Of course, this is well known for finite graphs.

In the last section of the paper we show that the vertex set of any graph can be endowed with a well-ordering which has a certain compactness property with respect to edge-separation, in the sense that given any (order-)bounded subset \(X \subseteq V(G)\) and any upper bound \(u\), if \(X\) cannot be separated from \(u\) by the removal of a finite number of edges, then the same is true for some finite subset of \(X\). In fact, we prove a similar statement, where the removal of a finite number of edges is replaced by the removal of fewer than \(\alpha\) edges, \(\alpha\) being an arbitrary regular cardinal. This result provides an interesting tool if one wishes to make a recursive construction on uncountable graphs and does not want the first steps of the construction to interfere “too much” with the rest.

2 Definitions and preliminaries

For the purposes of this paper, we assume all graphs to be unoriented, without loops or multiple edges unless otherwise stated. The symbol \(G\) will always denote a graph. A circuit is a 2-regular connected graph and a cycle is a finite circuit. A block of \(G\) is a 2-vertex-connected subgraph of \(G\) which is maximal with respect to inclusion; in particular a subgraph consisting of a bridge or a loop is a block. If \(L \subseteq E(G)\) then \(G \setminus L\) denotes the graph obtained from \(G\) by removing all edges in
If $X \subseteq V(G)$ then $G[X]$ denotes the induced subgraph of $G$ on $X$. If $x \in V(G)$ and $A, B$ denote subgraphs of $G$, we write $G - x = G[V(G) \setminus x]$, $G - A = G[V(G) \setminus V(A)], G \setminus A = G \setminus E(A)$ and $[A, B]_G$ denotes the set of edges of $G$ which join vertices of $A$ to vertices of $B$. When no confusion is likely we shall write $\overline{A}$ for $G - A$.

A cut of $G$ is a set of edges of the form $[A, \overline{A}]_G$. Unless otherwise stated, $A$ will be an induced subgraph of $G$. An odd (resp. even) cut is a cut whose cardinality is odd (resp. even). A bond is non-empty cut which is a minimal with respect to inclusion. Observe that a cut $[A, \overline{A}]_G$ of a connected graph $G$ is a bond if and only if both $A$ and $\overline{A}$ are connected.

**Remark 1** A cut $[A, \overline{A}]_G$ of a (connected or disconnected) graph is the union of a family of edge-disjoint bonds. It is easy to see that if $A$ or $\overline{A}$ is connected then the family is unique. If both $A$ and $\overline{A}$ are disconnected then the uniqueness does not hold, as illustrated by the example of Figure 1, where $[A, \overline{A}]_G$ is the union of the three bonds which consist respectively of the set of edges incident with each of the vertices of $A$, and also the union of the bonds symmetrically defined with respect to the three vertices of $\overline{A}$. In general, given a cut $[A, \overline{A}]_G$ of $G$, we can easily construct a suitable family $\mathcal{F}$ of bonds as follows: let $(A_i)_{i \in I}$ be the set of all components of the induced subgraph $A$; for each $i \in I$, let $\mathcal{F}_i$ be the unique family of bonds of $G$ whose union is $[A_i, \overline{A}_i]_G$, and then put $\mathcal{F} := \bigcup_{i \in I} \mathcal{F}_i$.

![Figure 1](image-url)

**Remark 2** Each bond of $G$ is contained in some block of $G$. To see this, suppose that $x$ is a cut-vertex of $G$ and $[A, \overline{A}]_G$ is a bond of $G$. If $x \in V(A)$ then the connected subgraph $\overline{A}$ of $G - x$ must be contained in a single component of $G - x$ and so $x$ cannot separate edges of $[A, \overline{A}]_G$; and a similar argument applies if $x \in V(\overline{A})$. 

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For any two distinct vertices \( x, y \in V(G) \), we denote by \( \gamma_G(x, y) \) the edge-connectivity between \( x \) and \( y \). By the weak version of Menger’s Theorem, \( \gamma_G(x, y) \) can be equivalently defined as the maximal cardinality of a set of edge-disjoint \( xy \)-paths of \( G \) or as the minimal cardinality of a cut of \( G \) that separates \( x \) from \( y \). Thus, \( \gamma_G(x, y) = 0 \) if and only if \( x \) and \( y \) belong to different connected components of \( G \). Observe that, assuming that every vertex is \( \kappa \)-edge-connected to itself, \( \kappa \)-edge-connectivity, unlike \( \kappa \)-vertex-connectivity, induces an equivalence relation on \( V(G) \) since, for each cardinal \( \kappa \),

\[
\gamma_G(x, y) \geq \kappa \text{ and } \gamma_G(y, z) \geq \kappa \implies \gamma_G(x, z) \geq \kappa.
\]

The equivalence classes of this relation are called the \( \kappa \)-edge-connectivity classes or simply \( \kappa \)-classes of \( G \). A graph that has exactly one \( \kappa \)-class is said to be \( \kappa \)-edge-connected.

A decomposition of \( G \) is an equivalence relation on \( E(G) \) such that the subgraph induced by the edges of any equivalence class is connected. The subgraphs induced in this way are called the fragments of the decomposition. Thus, a decomposition of \( G \) may be considered as a family of edge-disjoint connected subgraphs of \( G \) whose union is the graph \( G \) minus its isolated vertices. Among the most frequently studied decompositions are decompositions whose fragments are cycles (i.e., cycle decompositions) and decompositions whose fragments are cycles, rays or double rays. For results on the existence of such decompositions for infinite graphs, see Nash-Williams [4], Sabidussi [5], Thomassen [7], or Laviolette [2] and [3]. The main theorem of the present paper relies on what we will refer to as Nash-Williams’s Theorem:

**Theorem (Nash-Williams [4])** A graph has a cycle decomposition if and only if it does not contain any odd cut.

A decomposition whose fragments are all \( \kappa \)-edge-connected for some (finite or infinite) cardinal \( \kappa \), is said to be \( \kappa \)-edge-connected, and a decomposition whose fragments are all of cardinality less than or equal to \( \alpha \) for some infinite cardinal \( \alpha \), is called an \( \alpha \)-decomposition. Throughout, \( \alpha \) will denote an infinite cardinal, and \( \alpha^+ \) will denote the successor cardinal of \( \alpha \).

In this paper we look for decompositions whose fragments inherit the edge-connectivity of the graph up to a given cardinal. More precisely, we consider the following type of decompositions:

**Definition 1** An \( \alpha \)-decomposition \( \Delta \) of \( G \) is said to be bond-faithful if

(i) any bond of \( G \) of cardinality \( \leq \alpha \) is contained in some fragment of \( \Delta \);
(ii) any bond of cardinality $< \alpha$ of a fragment of $\Delta$ is also a bond in $G$.

In a bond-faithful $\alpha$-decomposition $\Delta$ of $G$, any bond $B$ of cardinality $\leq \alpha$ of $G$ is by (i), contained in some fragment $H$ and hence is a cut of $H$. Moreover, if $|B| < \alpha$, then this cut is always a bond of $H$ since otherwise there is a bond $B'$ of $H$, strictly contained in $B$, which because of (ii), must also be a bond of $G$, contradicting the fact that $B$ is a bond of $G$. Hence, the following properties are always satisfied for any set of edges $B \subseteq E(G)$:

1. if $|B| < \alpha$, then $B$ is a bond of $G$ if and only if it is a bond of some fragment of $\Delta$;
2. if $|B| = \alpha$, and $B$ is a bond of $G$, then $B$ is a cut of some fragment of $\Delta$;
3. if $|B| > \alpha$, and $B$ is a bond of $G$, then in any fragment $H$ containing edges of $B$, $B \cap E(H)$ is a cut of $H$ of cardinality $\alpha (= |H|)$.

Note, moreover, that since a cut is an edge-disjoint union of bonds, and because of condition (i) of the definition of bond-faithfulness, we can equivalently replace condition (ii) of that definition by:

(ii') any cut of cardinality $< \alpha$ of a fragment of $\Delta$ is also a cut in $G$.

A fundamental property of bond-faithful $\alpha$-decompositions, relating the local edge-connectivities of $G$ to those of the fragments of the decomposition, is expressed in the following proposition.

**Proposition 1** If $H$ is a fragment of a bond-faithful $\alpha$-decomposition of $G$ and $x$, $y$ any two vertices of $H$ then

$$\gamma_H(x, y) = \min\{\alpha, \gamma_G(x, y)\}$$

**Proof** Since $H \subseteq G$ we must have $\gamma_H(x, y) \leq \gamma_G(x, y)$. Hence if $\gamma_H(x, y) = \alpha$, there is nothing to show. On the other hand, if $\gamma_H(x, y) = \beta < \alpha$, then there exists a bond of $H$ of cardinality $\beta$ separating $x$ and $y$. By property (ii) of a bond-faithful $\alpha$-decomposition this implies that

$$\gamma_G(x, y) \leq \beta = \gamma_H(x, y) \leq \gamma_G(x, y)$$

\[\square\]
Remark 3  It follows from Proposition 1 that if $G$ is $\beta$-edge-connected, where $\beta \leq \alpha$, then every fragment of a bond-faithful $\alpha$-decomposition of $G$ is likewise $\beta$-edge-connected.

Since a decomposition of $G$ is an equivalence relation on $E(G)$ we have the following natural partial order on decompositions of a graph $G$.

**Definition 2** A decomposition $\Delta_2$ is *coarser* than $\Delta_1$ (denoted by $\Delta_2 \succeq \Delta_1$) if each fragment of $\Delta_1$ is contained in some fragment of $\Delta_2$.

With respect to this order, any (finite or infinite) family of decompositions has a supremum and an infimum. Since fragments have to be connected, the infimum does not always coincide with the infimum in the set of all equivalence relations. However, for the supremum (denoted by $\bigsqcup_{i \in I} \Delta_i$), the “connected” supremum coincides with the equivalence-supremum, as stated in the next lemma.

**Lemma 1** Let $(\Delta_i)_{i \in I}$ be a family of decompositions of a graph $G$. Then $\bigsqcup_{i \in I} \Delta_i$ is the transitive closure of the union of the equivalence relations $\Delta_i$.

**Proof** Since the transitive closure $\Delta$ of the union of the $\Delta_i$’s is already the supremum of the $\Delta_i$’s in the set of all equivalence relations on $E(G)$, one only has to show that every $\Delta$-equivalence class edge-induces a connected graph. This is straightforward and left to the reader. \qed

The supremum respects $\alpha$-decompositions and even preserves bond-faithfulness in a strong way.

**Lemma 2** Let $(\Delta_i)_{i \in I}$ be a family of $\alpha$-decompositions of $G$. If $|I| \leq \alpha$, then $\Delta = \bigsqcup_{i \in I} \Delta_i$ is an $\alpha$-decomposition; moreover, if the family contains at least one bond-faithful $\alpha$-decomposition, then $\Delta$ will also be bond-faithful.

**Proof** The first assertion follows from the fact that any fragment of $\Delta$ is the union of at most $\alpha$ fragments all of cardinality at most $\alpha$. Suppose now that the family contains a bond-faithful $\alpha$-decomposition $\Delta_0$. Then since $\Delta_0 \preceq \Delta$, any bond of $G$ of cardinality $\leq \alpha$ is contained in a fragment of $\Delta$. Moreover, if $B$ is a bond of a fragment $H$ of $\Delta$ of cardinality $< \alpha$, then for any edge $e \in B$ the intersection of $B$ with the fragment $H_0$ of $\Delta_0$ containing $e$ is a cut of $H_0$. Hence $B$ contains a bond $B_0$ of $H_0$. Since $\Delta_0$ is bond-faithful, $B_0$ is a bond of $G$. Since $B_0$ is a bond of $G$ and $B_0 \subseteq B \subseteq E(H)$, it follows that $B_0$ is a non-empty cut of $H$ contained in the bond $B$ of $H$ and so $B = B_0$, which is a bond of $G$. \qed
3 \( \omega \)-covers and 2-edge-connected decompositions

Given a cardinal \( \kappa \), a \( \kappa \)-cover of a graph \( G \) is a family \( (H_i)_{i \in I} \) of subgraphs of \( G \) such that each edge of \( G \) belongs to exactly \( \kappa \) members of the family. Hence a decomposition is a 1-cover with all members connected. The case which has received the most attention is \( \kappa = 2 \) with Seymour’s Double Cover Conjecture, which says that every 2-edge-connected graph admits a cycle 2-cover (i.e. a 2-cover all of whose members are cycles); see Seymour [6] or Bondy [1] for a survey. The following result is a (substantial) weakening of that conjecture.

**Theorem 1** Every 2-edge-connected graph has a cycle \( \omega \)-cover.

**Proof** Let \( x_0 \in V(G) \) and for each \( i > 0 \), let \( D_i \) be the set of edges of a 2-edge-connected graph \( G \) having one endpoint at distance \( i - 1 \) from \( x_0 \) and the other at distance \( i \). Let \( D_0 \) be the set of edges of \( G \) whose endpoints are at the same distance from \( x_0 \). Note that the \( D_i \)'s form a partition of \( E(G) \) into possibly empty sets and that for \( i \geq 1 \),

\[
D_i = [A_i, \overline{A_i}], \quad \text{where} \quad A_i = \{y \in V(G) : \text{dist}_G(x_0, y) \leq i - 1\}.
\]

We will now construct for each \( i \geq 0 \) a family \( F_i \) of cycles of \( G \) such that each edge of \( D_i \) belongs to at least one cycle of \( F_i \), and such that no edge of \( G \) belongs to more than \( \omega \) cycles of \( F_i \). To obtain \( F_0 \) (the simplest case) we proceed as follows. Form a multigraph \( G_0 \) by replacing each edge in \( G \setminus D_0 \) by \( \omega \) edges having the same endpoints. Note that \( G_0 \) is \( \omega \)-edge-connected since for any \( x \in V(G_0) (= V(G)) \) no edge of an \( x_0x \)-geodesic will belong to \( D_0 \); in other words, all edges of the geodesic will have been duplicated \( \omega \) times. Hence \( G_0 \) has no finite cut and therefore no odd cut, implying by Nash-Williams’s Theorem stated in Section 2 that \( G_0 \) has a decomposition into cycles, say \( \Delta_0 \). Any cycle of \( \Delta_0 \) canonically induces either a cycle in \( G \) or an edge in \( E(G) \setminus D_0 \), the latter case occurring only if the cycle of \( \Delta_0 \) is of length 2. Let \( F_0 \) be the family of all the cycles of \( G \) canonically induced by the cycles of \( \Delta_0 \). Then \( F_0 \) will have the desired properties since any edge in \( D_0 \) must belong to exactly one cycle in \( \Delta_0 \) and there are at most \( \omega \) cycles of \( F_0 \) that may contain a given edge.

Let us now construct \( F_i \) for \( i > 0 \). Since \( D_i \) is a cut of \( G \), it is the disjoint union of bonds (say \( D_i = \bigcup_{j \in J_i} B_{ij} \)). Given \( j \in J_i \), fix two arbitrary distinct edges \( e_1^j \) and \( e_2^j \) of \( B_{ij} \) (note that \( |B_{ij}| \geq 2 \) since hypothesis \( G \) is 2-edge-connected). In the same way as in the construction of \( G_0 \), let us construct \( G_i^k, k = 1, 2 \), by replacing in \( G \) each edge of \( E(G) \setminus D_i \) and each \( e_k^j \) (\( j \in J_i \)) by \( \omega \) edges having the same endpoints. Note that the \( G_i^k \)'s, \( i > 0, k = 1, 2 \), are all \( \omega \)-edge-connected.
since $V(G^k_i) = V(G)$ and the edges of $G$ which are being $\omega$-duplicated (i.e., the edges in $E(G) \setminus D_i \cup \{e_j^k : j \in J_i\}$ form a connected spanning subgraph of $G$.

Hence as we have done for $F_0$, we can construct two families of cycles $F^k_i$ ($k = 1, 2$) of $G$, obtained from a cycle decomposition of $G^k_i$, such that any edge of $G$ belongs to at most $\omega$ cycles of $F^k_i$ and any edge of $D_i \setminus \{e_j^k : j \in J_i\}$ belongs to at least one cycle of $F^k_i$. Since $\{e_j^1 : j \in J_i\}$ is disjoint from $\{e_j^2 : j \in J_i\}$, $F_i := F^1_i \cup F^2_i$ will have the desired two properties (a cycle is allowed to appear more than once in the family).

Finally it is easy to see that the family consisting of $\omega$ copies of every cycle in $\bigcup_{i \geq 0} F_i$ is an $\omega$-cover of $G$. □

The theorem of Nash-Williams used in this proof is based on a highly non-trivial transfinite induction. However, as will be seen later, Theorem 1 implies Corollary 2, which allows a reduction of the proof of Nash-Williams’s Theorem to the countable case which is easy to handle (see Remark 4). Hence any direct proof of Theorem 1 will give rise to a direct proof of Nash-Williams’s Theorem. Moreover, Theorem 1 gives some partial answer to the Cycle 2-Cover Conjecture in the infinite case.

**Corollary 1** Every bridgeless graph admits a 2-edge-connected $\omega$-decomposition.

**Proof** Let $G$ be such a graph. We may clearly suppose that $G$ is connected, i.e. 2-edge-connected. Let $\Phi$ be a cycle $\omega$-cover of $G$ given by Theorem 1 and $\Delta$ the equivalence relation defined as the transitive closure of the relation $\Theta$ on $E(G)$, where $e \Theta e'$ if and only if $\Phi$ contains a cycle containing both $e$ and $e'$.

**Claim:** $\Delta$ is a 2-edge-connected $\omega$-decomposition. Let $H$ be a fragment of $\Delta$.

1. $H$ is connected, since for any two edges $e, e' \in E(H)$ there exist $e_1, \ldots, e_n \in E(H)$ such that $e = e_1, e' = e_n$ and $e_i \Theta e_{i+1}$ for any $i = 1, 2, \ldots, n - 1$. Let $C_i \in \Phi$ be a cycle containing both $e_i$ and $e_{i+1}$ and note that $\bigcup_{i=1}^n C_i$ is a connected subgraph of $H$ containing $e$ and $e'$.

2. $H$ is trivially 2-edge-connected since any edge $e \in E(H)$ is contained in a cycle of $\Phi$ which belongs to $H$.

3. $H$ is at most countable since any edge $e \in E(H)$ is $\Theta$-related to at most $\omega$ other edges, and $\Delta$ is the transitive closure of $\Theta$. □

Observe that Corollary 1 implies that the Cycle 2-Cover Conjecture is true for graphs of arbitrary cardinality provided it is true in the countable case. We even have the following stronger result:
Proposition 2 If every 3-regular bridgeless graph has a cycle 2-cover, then so does every bridgeless graph.

This result is already known in the finite case.

Proof Let $G$ be any bridgeless graph. Without loss of generality we may suppose that $G$ is connected, without vertices of degree 2 and, by Corollary 1, countable. Let $\hat{G}$ be the 3-regular graph obtained from $G$ in the following way: for each $x \in V(G)$ let $C_x$ be a cycle of length $\deg_G(x)$ if $x$ is of finite degree, or a double ray (i.e., infinite circuit) otherwise (where the $C_x$’s are pairwise disjoint and disjoint from $G$). Let $\phi_x$ be any bijection from $V(C_x)$ to the set of edges of $G$ incident with $x$. Then,

\begin{align*}
V(\hat{G}) &:= \bigcup_{x \in V(G)} V(C_x), \\
E(\hat{G}) &:= \bigcup_{x \in V(G)} E(C_x) \cup \\{ [a, b] : a \in V(C_y), b \in V(C_z), y \neq z \text{ and } \phi_y(a) = \phi_z(b) \}.
\end{align*}

See Figure 2 for an example.

![Figure 2](image)

Clearly $\hat{G}$ is 3-regular. Moreover, since there is a canonical bijection from the edges of $\hat{G}$ not belonging to the $C_x$’s to $E(G)$, it is easy to see that any cycle of $\hat{G}$ different from the $C_x$’s gives rise in $G$ to a finite eulerian graph (which is a union of edge-disjoint cycles), and therefore any cycle 2-cover of $\hat{G}$ will induce a cycle 2-cover in $G$. □

4 Bond-faithful $\alpha$-decompositions

The aim of this section is to show that every graph has a bond-faithful $\omega$-decomposition, and that, assuming the Generalized Continuum Hypothesis (GCH), every graph has a bond-faithful $\alpha$-decomposition, for any infinite cardinal $\alpha$.

Lemma 3 Let $\alpha$ be a regular infinite cardinal, and $\Delta_0$ be an $\alpha$-decomposition of $G$. Assume GCH if $\alpha > \omega$. Then there exists an $\alpha$-decomposition $\Delta$ which is coarser than $\Delta_0$ and has the property that for any fragment $H$ of $\Delta_0$, the only bonds of $H$ of cardinality $< \alpha$ which are bonds of the corresponding fragment of $\Delta$ are those which are bonds of $G$. 

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Thus Δ “purifies” the fragments of $\Delta_0$ of all bonds of cardinality $< \alpha$ that are not bonds in $G$.

**Proof** For each fragment $H$ of $\Delta_0$, let $(B^H_\beta)_{\beta \in \gamma_H}$ be any well-ordering of the set of all the bonds of cardinality $< \alpha$ of $H$ that are not bonds of $G$. Then for each $\beta \in \gamma_H$, fix an edge $e^K_\beta$ in $B^K_\beta$. Since $|E(H)| \leq \alpha$, by the Generalized Continuum Hypothesis, $H$ has at most $\alpha$ bonds of cardinality $< \alpha$. Thus $\gamma_H \leq \alpha$ for any $H$. Let

$$G_\beta := G \setminus \bigcup \{B^H_\beta \setminus e^K_\beta : H \text{ is a fragment of } \Delta_0 \text{ and } \gamma_H > \beta\}$$

for any $\beta < \alpha$. Given any fragment $K$ of $\Delta_0$ and any $\beta < \gamma_K$, $e^K_\beta$ is an edge of $G_\beta$ because the fragments of $\Delta_0$ are pairwise edge-disjoint. We claim that $e^K_\beta$ is however not a bridge of $G_\beta$. Otherwise, $e^K_\beta$ will still be a bridge in $G \setminus (B^K_\beta \setminus e^K_\beta)$ because $G \setminus (B^K_\beta \setminus e^K_\beta)$ can be obtained from $G_\beta$ by putting back every $B^H_\beta \setminus e^K_\beta$ except $B^K_\beta \setminus e^K_\beta$, and because $H \setminus (B^H_\beta \setminus e^K_\beta)$ is a connected subgraph of $G$ for every fragment $H$ of $\Delta_0$. Hence $B^K_\beta$ will be a cut of $G$, and therefore it will be a bond of $G$, a contradiction.

Now, for each $\beta < \alpha$, apply Corollary 1 and choose a 2-edge-connected $\omega$-decomposition $\Gamma_\beta$ of $G_\beta \setminus \{e \in E(G_\beta) : e \text{ is a bridge of } G_\beta\}$. Then let $\Phi_\beta$ be the $\omega$-decomposition of $G$ obtained from $\Gamma_\beta$ by adding every bridge of $G_\beta$ and every edge of $G \setminus G_\beta$ as an equivalence class of one element. Moreover, for each edge $e^K_\beta$, fix a cycle $C^K_\beta$ that contains $e^K_\beta$ and is contained in the fragment of $\Phi_\beta$ that contains $e^K_\beta$. Hence $B^K_\beta \cap E(C^K_\beta) = \{e^K_\beta\}$ for any fragment $H$ of $\Delta_0$ and any $\beta < \gamma_H$.

Let us show that $\Delta := \Delta_0 \cup (\bigcup_{\beta < \alpha} \Phi_\beta)$, is the desired $\alpha$-decomposition. Clearly, $\Delta_0 \preceq \Delta$, and it follows from Lemma 2 that $\Delta$ is an $\alpha$-decomposition. Denote by $L_H$ the fragment of $\Delta$ that contains $H$ (and hence all the $e^K_\beta$’s). Since $C^K_\beta$ is contained in $L_H$ for any $H$, $G \setminus (C^K_\beta \setminus e^K_\beta)$ is therefore a path (edge-disjoint from $B^K_\beta$) that connects (in $L_H$) the two components which are separated by $B^K_\beta$ in $H$. Thus no $B^K_\beta$ can be a bond of $L_H$. \qed

Applying the preceding lemma $\alpha$ times we will obtain an $\alpha$-decomposition satisfying condition (ii) of the bond-faithfulness definition. This is the content of the following corollary.

**Corollary 2** Let $\alpha$ be a regular infinite cardinal, and $\Delta_0$ be an $\alpha$-decomposition of $G$. Then there exists an $\alpha$-decomposition $\Delta$ such that $\Delta_0 \preceq \Delta$ and any bond of cardinality less than $\alpha$ of a fragment of $\Delta$ is also a bond in $G$.

**Proof** By Lemma 2 we can inductively construct an increasing sequence $\{\Delta_\beta\}_{\beta < \alpha}$ of $\alpha$-decompositions as follows:

- $\Delta_0$ is the decomposition given in the hypothesis;
- $\Delta_{\beta+1}$ is an $\alpha$-decomposition such that $\Delta_\beta \preceq \Delta_{\beta+1}$ and has the property of Lemma 3 with $\Delta_0, \Delta$ replaced by $\Delta_{\beta}, \Delta_{\beta+1}$ respectively;
- $\Delta_\lambda = \bigcup_{\beta < \lambda} \Delta_\beta$, if $\lambda$ is a limit ordinal.

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We claim that $\Delta = \bigvee_{\beta<\alpha} \Delta_\beta$ is an $\alpha$-decomposition having the desired properties. First note that $\Delta_0 \leq \Delta$ and that, by Lemma 2, $\Delta$ is an $\alpha$-decomposition. Now, by way of contradiction, let $B$ be any bond of cardinality $\leq \alpha$ of a fragment $H$ of $\Delta$ which is not a bond of $G$. If $K$ is the component of $G$ that contains $H$, then $K \setminus B$ is still a connected graph, and hence no subset of $B$ can be a bond of $G$. Fix an edge $e \in B$ and, for any ordinal $\beta < \alpha$, denote by $H_\beta$ the fragment of $\Delta_\beta$ that contains $e$. It is easy to see that $(H_\beta)_{\beta<\alpha}$ is a nested sequence of subgraphs of $G$ whose union is $H$, and that $B \cap E(H_\beta)$ is a cut of $H_\beta$. Cuts being edge-disjoint unions of bonds, there is a bond $[A_\beta, A_\beta]_{H_\beta}$ of $H_\beta$ that is contained in $B$. If no subset of $B \cap E(H_\beta)$ is a bond of $G$, $[A_\beta, A_\beta]_{H_\beta}$ is not a bond of $H_{\beta+1}$. This and the fact that $H_{\beta+1} \setminus [A_\beta, A_\beta]_{H_\beta}$ is composed of exactly two connected components (viz. $A_\beta$ and $A_\beta$), implies that $H_{\beta+1} \setminus [A_\beta, A_\beta]_{H_\beta}$ is connected. Hence there exists an $A_\beta A_\beta$-path that is totally contained in $H_{\beta+1} \setminus H_\beta$, and therefore

$$B \cap (E(H_{\beta+1}) \setminus E(H_{\beta})) \neq \emptyset$$

for any $\beta < \alpha$.

It follows that $B$ is of cardinality $\geq \alpha$, a contradiction. \hfill $\square$

**Remark 4** Let $G$ be a graph without any odd cut. Clearly, any decomposition $\Delta$ of $G$ given by Corollary 2 (with $\alpha := \omega$ and $\Delta_0$, the decomposition all of whose fragments are single edges) will only consist in countable fragments with no odd cut. Thus, as stated before, Corollary 2 allows a reduction of the proof of the Nash-William’s Theorem to the countable case.

Before proceeding to our main theorem we need one last result which shows that a vertex of “high” degree in a graph is either “highly” connected to some other vertex or is a cut-vertex.

**Theorem 2** Let $G$ be a connected graph (possibly with loops and multiple edges), $x \in V(G)$ and $\mu$ be a regular uncountable cardinal. If $\text{deg}_G(x) \geq \mu$, then $x$ is a cut-vertex of $G$ or is $\mu$-vertex-connected to some vertex $y \neq x$.

Here $\text{deg}_G(x)$ only counts the neighbors of $x$ and not the (possibly greater) number of incident edges. Note however that since $\mu$ is a regular uncountable cardinal, it is easy to show that the result still holds when we define $\text{deg}_G(x)$ as the number of incident edges, provided we consider as being $\mu$-vertex-connected any two vertices linked by a multiple of $\mu$ edges.

**Proof** Suppose that $x$ is not a cut-vertex of $G$. Hence $G - x$ is still connected; choose a spanning tree $T$ of $G - x$ and let $J$ be the union of all cycles of $T \cup A$, where $A$ is the subgraph of $G$ induced by all the edges incident with $x$. Since $T$ is a tree any cycle of $T \cup A$ must contain $x$. Hence $J$ is connected. Moreover, since $T$ is connected, any two edges $e_1, e_2$ of $G$ incident with $x$ must be contained in some cycle of $T \cup A$, implying that $A \subseteq J$ and that $J_1 = J - x$ is a tree.
We claim that some \( y \in V(J_1) \) has degree at least \( \mu \) in \( J_1 \). By way of contradiction, suppose this is not the case. Let \( u \) be any vertex of \( J_1 \). By a straightforward inductive argument one can show that the sets
\[
D_i := \{ v \in V(J_1) : \text{dist}_{J_1}(u, v) = i \}
\]
are all of cardinality less than \( \mu \) because \( \mu \) is regular and \( |D_i| \leq \sum_{v \in D_{i-1}} \deg_{J_1}(v) \) for any \( i > 0 \). This gives rise to a contradiction since \( V(J_1) \subseteq \bigcup_{i \in \omega} D_i \), \( |J_1| \geq \mu \) and \( \mu \) is a regular cardinal.

Note that \( J - y \) is connected because as already stated, every cycle of \( T \cup A \) must contain \( x \). However, since \( J_1 = J - x \) is a tree, \( J - \{x, y\} \) will break into at least \( \mu \) components, and from each of these components together with \( x \) and \( y \) one can construct an \( xy \)-path. In this way we obtain at least \( \mu \) internally vertex-disjoint \( xy \)-paths. \( \Box \)

**Corollary 3** Let \( \alpha \) be any infinite cardinal. If a connected graph \( G \) (possibly with loops and multiple edges) contains no two distinct \( \alpha^+ \)-edge-connected vertices, then every block of \( G \) has cardinality at most \( \alpha \).

**Proof** By way of contradiction, suppose \( B \) is a block of \( G \) of cardinality \( > \alpha \). Since \( \alpha^+ \) (the successor cardinal of \( \alpha \)) is a regular uncountable cardinal, some vertex must have degree at least \( \alpha^+ \) in \( B \) and so, by Theorem 2 either \( B \) has a cut-vertex (contradicting the definition of a block) or two distinct vertices are \( \alpha^+ \)-vertex-connected in \( B \) and therefore \( \alpha^+ \)-edge-connected in \( G \) (contradicting the hypothesis). \( \Box \)

**Proposition 3** Let \( \alpha \) be an infinite cardinal and assume GCH if \( \alpha > \omega \). Then every graph has an \( \alpha \)-decomposition that satisfies the property (i) of the definition of bond-faithfulness.

**Proof** Clearly we may consider a connected graph \( G \). Moreover, we may assume that \( |G| \geq \alpha^+ \) since otherwise we can take the decomposition having \( G \) as its only fragment.

Let \( \sigma \) be the equivalence relation on \( V(G) \) induced by \( \alpha^+ \)-edge-connectivity, i.e.,
\[
x \sigma y \text{ if and only if } x = y \text{ or } \gamma_G(x, y) \geq \alpha^+.
\]

Let \( G/\sigma \) be the quotient graph modulo \( \sigma \), in other words, the graph obtained from \( G \) by identifying the vertices of each \( \sigma \)-class without identifying any edge. Thus \( G/\sigma \) may have loops and multiple edges. Since there is a canonical bijection between \( E(G) \) and \( E(G/\sigma) \), we will suppose for convenience that \( E(G) = E(G/\sigma) \). We shall also use the following notation: given a subgraph \( H \) of \( G/\sigma \), we denote by \( \hat{H} \) the lifted subgraph of \( G \) corresponding to \( H \) (i.e., the subgraph formed by the edges of \( H \), considered as edges of \( G \), together with their incident vertices).

By Corollary 3, the blocks of \( G/\sigma \) are of cardinality \( \leq \alpha \). Hence by Remark 2 so are the bonds of \( G/\sigma \). Since these bonds are also bonds of \( G \) and since a bond of \( G \) of
cardinality $\leq \alpha$ cannot separate two $\alpha^+$-edge-connected vertices, it follows that the bonds of $G/\sigma$ are exactly the bonds of $G$ of cardinality $\leq \alpha$.

Let $\Delta_1$ be the decomposition of $G/\sigma$ whose fragments are its blocks. Clearly $\Delta_1$ is a bond-faithful $\alpha$-decomposition of $G/\sigma$ but unfortunately not necessarily a decomposition of $G$, because the subgraph of $G$ induced by the edges of a block of $G/\sigma$ is not necessarily connected.

The existence of such a decomposition of $G$ is a consequence of the following:

**Claim:** From the set $(H_i)_{i \in I}$ of all blocks of $G/\sigma$, one can construct a family $(K_i)_{i \in I}$ of connected subgraphs of $G$ such that

1. $\hat{H}_i \subseteq K_i$ for any $i \in I$;
2. $|K_i| \leq \alpha$ for any $i \in I$;
3. each edge $e \in E(G)$ belongs to at most finitely many different $K_i$’s.

Indeed, assuming the claim to be true, it is easy to see that a suitable $\alpha$-decomposition of $G$ is the equivalence relation defined as the transitive closure of the relation $\Theta$ given by:

$$e \Theta e' \iff e, e' \in E(K_i) \text{ for some } i \in I.$$

**Proof of the claim:** Suppose $0 \in I$ and consider the partial order $\leq$ on the index set $I$ arising from the block-cutpoint tree of $G/\sigma$, i.e.,

$$i < j \iff i \neq j \text{ and some (and hence any) path of } G/\sigma \text{ joining a vertex of } H_0 \text{ to a vertex of } H_j \text{ contains an edge of } H_i.$$

(See Figure 3 for an example.)

![Figure 3: In this example, $i < j$.](image-url)
We have chosen to define strict inequality on $I$ because in the case where $H_i$ is a loop, no path of $G/\sigma$ joining a vertex of $H_0$ to a vertex of $H_i$ contains an edge of $H_i$, and even if $H_i$ is not a loop, then some but not all of those paths contain such an edge.

For each $i \in I$ let

$$L_i := \bigcup_{j \geq i} H_j.$$ 

Since any $i \in I$ has only finitely many predecessors in the order $\leq$ defined above, it follows that any edge $e \in E(G/\sigma)$ belongs to at most finitely many $L_j$’s, namely those for which $j \leq i$, where $i_e$ is the subscript of the unique $H_i$ that contains $e$.

Clearly $L_i$ is connected: let us prove that so is $\tilde{L}_i$. If $i = 0$, then $\tilde{L}_i = G$ which is connected by assumption. If $H_i$ is a loop, then $i$ is $\leq$-maximal which implies that $H_i = L_i$ and hence that $\tilde{L}_i$ is connected (indeed, a single edge). If $i \neq 0$ and $H_i$ is not a loop, then let $q_i$ be the unique cut-vertex of $G/\sigma$ belonging to $H_i$ that separates the edges of $L_i$ from those of $H_0$. Observe that any two $\sigma$-equivalent vertices $x, y \in V(\tilde{L}_i) \subseteq V(G)$, which do not belong to the $\sigma$-class $Q_i$ of $G$ corresponding to $q_i$, are connected in $G$ by $\alpha^+$ edge-disjoint paths. At most $\alpha$ of these paths can meet $Q_i$ because otherwise $x$ and $y$ would belong to $Q_i$. Thus, $x$ and $y$ are connected (in fact $\alpha^+$-edge-connected) in $\tilde{L}_i$. This, together with the fact that $L_i - q_i$ is connected, implies that $\tilde{L}_i - Q_i$ (the lifted graph corresponding to $L_i - q_i$) is connected. Hence if $\tilde{L}_i$ is not connected, all but one of its components (namely the one that contains $\tilde{L}_i - Q_i$) have all their vertices in $Q_i$. Any such component corresponds in $L_i$ to a union of loops at $q_i$. Being blocks contained in $L_i$, these loops are among the $H_j$’s with $j \geq i$. But by the definition of the order, any loop at $q_i$ is either $H_i$ itself (which is excluded by assumption) or has a subscript which is incomparable with $i$. Thus we have reach a contradiction, i.e., $\tilde{L}_i$ is connected.

It is not hard to see (but not needed for the rest of the proof) that the $\tilde{L}_i$’s satisfy conditions (1) and (3). Their cardinality, however, may exceed $\alpha$. To overcome this difficulty, choose a spanning tree $T_i$ of $\tilde{L}_i$ ($i \in I$) and define $K_i$ to be the union of $\tilde{H}_i$ and all paths in $T_i$ that connect two vertices of $\tilde{H}_i$. Clearly, $K_i$ is a connected subgraph of $\tilde{L}_i$ (and hence of $G$). To finish the proof of the claim, let us show that the family $(K_i)_{i \in I}$ has the required properties (1), (2), (3).

1. $\tilde{H}_i \subseteq K_i$ is trivially true for any $i \in I$.
2. $|E(K_i)| \leq \alpha$ for any $i \in I$, because so is $|E(H_i)|$ which is equal to $|E(\tilde{H}_i)|$, and because $K_i$ is the union of $\tilde{H}_i$ and at most $\alpha^2$ paths of $T_i$.
3. This is a consequence of the fact that any edge $e \in E(G)$ can belong to at most finitely many $\tilde{L}_j$’s, because as has been shown earlier $e$ (viewed as an edge of $G/\sigma$) can belong to at most finitely many $L_j$’s.

The following is our main theorem.

**Theorem 3** Every graph has a bond-faithful $\omega$-decomposition, and with the assumption of GCH, every graph has a bond-faithful $\alpha$-decomposition for any infinite cardinal $\alpha$. 

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Proof. Case 1: $\alpha$ is regular. This is immediate from Proposition 3 and Corollary 2.

Case 2: $\alpha$ is singular. Let $G$ be a graph, and for each regular infinite cardinal $\beta < \alpha$, let $\Delta_\beta$ be a bond-faithful $\beta$-decomposition of $G$. Apply Proposition 3 to choose an $\alpha$-decomposition $\Delta_0$ of $G$ that satisfies Property (i) of the definition of bond-faithfulness. Finally, put $\Delta := \Delta_0 \lor \bigvee_\beta \Delta_\beta$.

We claim that $\Delta$ is a bond-faithful $\alpha$-decomposition. By Lemma 2, $\Delta$ is an $\alpha$-decomposition because any $\beta$-decomposition with $\beta < \alpha$ is also an $\alpha$-decomposition. Moreover, since $\Delta_0 \preceq \Delta$, then $\Delta$ also satisfies Property (i) of the definition of bond-faithfulness. Thus, it remains to show that $\Delta$ satisfies Property (ii).

Let $B$ be any bond of cardinality $\beta < \alpha$ of any fragment $H$ of $\Delta$. Choose any regular infinite cardinal $\gamma$ such that $|B| < \gamma < \alpha$, and any fragment $H_\gamma$ of $\Delta_\gamma$ that edge-intersects $B$. It follows from $E(H_\gamma) \subseteq E(H)$, that $B \cap E(H_\gamma)$ is a non-empty cut of $H_\gamma$. Hence, cuts being edge-disjoint unions of bonds, there therefore exists a bond $C$ of $H_\gamma$ such that $C \subseteq B \cap E(H_\gamma)$. We have that $C$ is a bond of $G$, because $\Delta_\gamma$ is a bond-faithful $\gamma$-decomposition, and because $|C| \leq |B| < \gamma$. Thus $C$ is a cut of $H$. As it is in addition, a bond of $H_\gamma \subseteq H$ it must be a bond of $H$. Since $B$ is also a bond of $H$ and since $C \subseteq B$, we must have $C = B$. Thus $B$ is a bond of $G$, and we are done.

Theorem 3 implies the following apparently stronger result; here we distinguish the case where $\alpha = \omega$ (that will be used in [3]) to the one where it is uncountable.

In the countable case:

Theorem 4. Let $(H_i)_{i \in I}$ be a family of edge-disjoint connected countable subgraphs of $G$. Then $G$ has a bond-faithful $\omega$-decomposition $\Delta$ such that each $H_i$ and each non-isolated vertex of degree $\leq \omega$ in $G$ is contained in one and only one fragment of $\Delta$.

In the uncountable case:

Theorem 5. Let $\alpha$ be an uncountable cardinal, and $(H_i)_{i \in I}$ be a family of edge-disjoint connected subgraphs of order $\leq \alpha$ of $G$. Then, assuming $GCH$, $G$ has a bond-faithful $\alpha$-decomposition $\Delta$ such that each $H_i$ and each non-isolated vertex of degree $\leq \alpha$ in $G$ is contained in one and only one fragment of $\Delta$.

Proof of Theorems 4 and 5. Let $\Delta_1$ be any bond-faithful $\alpha$-decomposition of $G$, $\Delta_2$ the $\alpha$-decomposition of $G$ whose fragments are the $H_i$’s and each of the edges of $G$ which do not belong to any $H_i$, and $\Delta_3$ the $\alpha$-decomposition which is the transitive and reflexive closure of the following binary relation $\Theta$:

$$e \Theta e' \iff \text{both } e, e' \text{ are incident to } x \text{ for some vertex } x \text{ of degree } \leq \alpha \text{ in } G.$$  

By Lemma 2, $\Delta := \Delta_1 \lor \Delta_2 \lor \Delta_3$ is the desired decomposition.
5 Other decompositions

Bond-faithful $\alpha$-decompositions provide a way of splitting a graph into edge-disjoint subgraphs, each of which preserves the “small” edge-connectivities of the original graph (i.e., not greater than $\alpha$). In this section, we will show that a graph can also be split into edge-disjoint subgraphs which preserve the “high” edge-connectivities of the original graph.

**Proposition 4** Every graph $G$ is the edge-disjoint union of two (not necessarily connected) spanning subgraphs, say $K$ and $L$, such that

\[ \gamma_K(x, y) = \gamma_L(x, y) = \gamma_G(x, y) \]

for each pair $x, y$ of infinitely edge-connected vertices of $G$.

**Proof** We leave it to the reader to show that this is true for countable graphs. So suppose $G$ is uncountable. By Theorem 3, there exists a bond-faithful $\omega$-decomposition $\Delta = (H_i)_{i \in I}$ of $G$. Since we assume the proposition to be proved in the countable case, and each $H_i$ is countable, $H_i$ is the union of two edge-disjoint subgraphs $K_i$ and $L_i$ such that any pair of vertices $x, y \in V(H_i)$ which are infinitely edge-connected in $H_i$ are also infinitely edge-connected in both $K_i$ and $L_i$. Let $K := \bigcup_{i \in I} K_i$ and $L := \bigcup_{i \in I} L_i$, and let us prove that they both preserve $\alpha$-edge-connectivity for any $\alpha \geq \omega$ or, in other words, that for any $x, y \in V(G)$ with $\gamma_G(x, y) = \alpha$, we have $\gamma_K(x, y) = \alpha$ and $\gamma_L(x, y) = \alpha$. Note that by symmetry, we only have to show that $\gamma_K(x, y) = \alpha$.

Take a set $P = (P_\beta)_{\beta < \alpha}$ of edge-disjoint $xy$-paths of $G$ and subdivide each $P_\beta$ into edge-disjoint consecutive subpaths $P^1_\beta, P^2_\beta, \ldots, P^{j_\beta}_\beta$ such that

- $x$ is an end-vertex of $P^1_\beta$ and $y$ of $P^{j_\beta}_\beta$,
- the end-edges of each $P^1_\beta$ belong to the same fragment of $\Delta$;
- no edge of $P^{j_\beta+1}_\beta \cup P^{j_\beta+2}_\beta \cup \ldots \cup P^{j_\beta}_\beta$ belongs to the fragment of $\Delta$ that contains the end-edges of $P^1_\beta$, for any $j$.

To finish the proof we will show that there exists a set $Q = (Q_\beta)_{\beta < \alpha}$ of edge-disjoint $xy$-paths of $K$ such that for each $\beta < \alpha$, $Q_\beta$ can be subdivided into $Q^1_\beta \cup Q^2_\beta \cup \ldots \cup Q^{j_\beta}_\beta$ such that

- $P^1_\beta$ and $Q^1_\beta$ have the same end-vertices;
- $Q^1_\beta$ is contained in $K_i$ where $H_i$ is the fragment of $\Delta$ that contains the two end-edges of $P^1_\beta$.

Such a family $Q$ exists if for each fragment $H_i$ of $\Delta$ the set $P_i$ of all the $P^1_\beta$’s whose end-edges belong to $H_i$ is in one-to-one correspondence with some set of edge-disjoint paths of $K_i$ such that each $P^1_\beta$ corresponds to a path having the same end-vertices. Since $H_i$, and hence $P_i$, is countable, we only have to show that the two end-vertices of each subpath in $P_i$ are infinitely edge-connected in $K_i$. By way of contradiction suppose there
exists some $P^j_β$ in $P_i$ whose end-vertices $u, v$ satisfy $γ_{K_i}(u, v) < ω$ and suppose that $j$ is the least integer for which there exists such a $P^j_β$. By the choice of $K_i$ we have $γ_{H_i}(u, v) < ω$, and so some finite bond $B$ of $H_i$ separates $u$ from $v$ in $H_i$. Since $Δ$ is bond-faithful, $B$ is also a bond of $G$. Moreover, since $P^j_{β+1} ∪ ... ∪ P^j_{β}$ is edge-disjoint from $H_i$, it is edge-disjoint from $B$, implying that $B$ not only separates $u$ from $v$ in $G$, but also $u$ from $y$. Thus, $γ_G(u, y) < ω$. On the other hand, $γ_G(x, u) ≥ ω$ by the minimality of $j$; therefore $x$ and $y$ cannot be infinitely edge-connected in $G$, a contradiction. □

**Proposition 5** Assuming GCH if $α > ω$, every $α$-edge-connected graph can be decomposed into $α$-edge-connected spanning fragments. 

**Proof** We leave it to the reader to show that this is true if $|G| = α$. For the case $|G| > α$, let $(H_i)_{i ∈ I}$ be a bond-faithful $α$-decomposition of $G$. Since $G$ is $α$-edge-connected, by Remark 3, each $H_i$ is also $α$-edge-connected. Being fragments of an $α$-decomposition, each $H_i$ is therefore of cardinality $α$. Decompose each $H_i$ into $α$-edge-connected fragments that are spanning in $H_i$, say $(H^β_i)_{β < α}$, and for each $β < α$ let

$$H^β := \bigcup_{i ∈ I} H^β_i.$$ 

It is easy to see that each $H^β$ is spanning in $G$ and $α$-edge-connected; thus $(H^β)_{β < α}$ is the desired decomposition. □

**Corollary 4** Assuming GCH if $α > ω$, every $α$-edge-connected graph contains at least $α$ edge-disjoint spanning trees.

For arbitrary (not necessarily $α$-edge-connected) graphs the preceding result still holds in the following form:

**Proposition 6** Assume GCH if $α > ω$, and let $W$ be an $α$-class of $G$. Then there exists a family $(T^β)_{β < α}$ of edge-disjoint trees of $G$ such that $W ⊆ V(T^β)$ for any $β < α$.

**Proof** Again we leave it to the reader to show that this is true if $|G| = α$. For the case $|G| > α$, let $Δ := (H_i)_{i ∈ I}$ be a bond-faithful $α$-decomposition of $G$. Observe that $H_i ∩ W$ is either an $α$-class of $H_i$ or empty. For each $H_i$, choose a family $(T^β_i)_{β < α}$ of edge-disjoint trees of $H_i$ such that $H_i ∩ W ⊆ V(T^β_i)$. Observe that $W ⊆ V(∪_{i ∈ I} T^β_i)$ and that if for each $β < α$, $W$ is contained in a single component of $∪_{i ∈ I} T^β_i$, we will be done by taking as $T^β_i$ any spanning tree of that component. Hence, to finish the proof, suppose by way of contradiction that there exist $x, y ∈ W$ that belong to different components of $∪_{i ∈ I} T^β_i$. Let $P$ be any $xy$-path of $G$ and
without loss of generality suppose $V(P) \cap W = \{x, y\}$. If $H_j$ is the fragment of $\Delta$ that contains the edge of $P$ which is incident to $x$ then $y \notin V(H_j)$; let $z$ be the last vertex of $P$ that belongs to $H_j$. Since $z \notin W$, $\gamma_{H_j}(x, z) < \alpha$ and hence $\gamma_{H_j}(x, z) < \alpha$, implying that there exists a bond $B$ of cardinality $< \alpha$ separating $x$ from $z$ in $G$, and therefore also separating $x$ from $y$, because the $zy$-path contained in $P$ is edge-disjoint from $H_j$ and hence from $B$, a contradiction to $\gamma_G(x, y) \geq \alpha$. 

\section{A special well-ordering on vertices of graphs}

Theorem 2 has the following consequence:

\textbf{Theorem 6} Let $G$ be a graph (possibly with loops and multiple edges) and $\alpha$ an uncountable regular cardinal. Then for any $\alpha$-class $X$ of $G$, $[X, X]_G$ is a union of bonds of cardinality less than $\alpha$.

\textbf{Proof} Consider $G/X$, the graph obtained from $G$ by identifying the vertices of $X$ and denote by $x$ the new vertex so obtained. If $[X, X]_G$ contains a bond of $G$ of cardinality $\geq \alpha$ then $G/X$ contains a block in which $x$ has degree $\geq \alpha$, contradicting Theorem 2 applied to that block. 

This result, interesting in its own right, also has the striking consequence that it is always possible to well-order the $\omega$-classes of a graph in such a way that the union of all the $\omega$-classes that precede any given one is separable from it by a finite cut. Since it is always possible to separate a finite set of $\omega$-classes from any other one, it is easy to construct such a well-ordering when there are at most countably many $\omega$-classes. The real problem occurs when there are uncountably many. The existence of such a well-ordering can be a very useful tool for constructions on infinite graphs.

The next theorem establishes this result and generalizes it to any infinite regular cardinal.

\textbf{Theorem 7} Let $\alpha$ be a regular infinite cardinal and $W$ the set of all $\alpha$-classes of $G$. Then there exists a well-ordering on $W$ (say $W = ([x_\delta])_{\delta < \beta}$) such that each $[x_\mu] \in W$ can be separated from $\bigcup_{\delta < \mu} [x_\delta]$ by a cut of cardinality $< \alpha$ of $G$.

\textbf{Proof} Case 1. $|G| \leq \alpha$. We claim that in this case any well-ordering $([x_\delta])_{\delta < \beta}$ of $W$ with $\beta \leq \alpha$ has the desired property.

Let $[x_\mu] \in W$ and for each $\delta < \mu$ let $[A_\delta, \overline{A_\delta}]_G$ be a cut of cardinality $< \alpha$ such that $[x_\delta] \subseteq V(A_\delta)$ and $[x_\mu] \subseteq V(\overline{A_\delta})$. Now observe that

$$B := \left[ \bigcup_{\delta < \mu} A_\delta, \bigcup_{\delta < \mu} A_\delta \right]_G$$

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is a cut separating \([x_\mu]\) from all the \([x_\delta]\)'s, \(\delta < \mu\), and moreover that \(|B| < \alpha\), because

\[ |B| \leq \sum_{\delta < \mu} |A_\delta, \overline{A_\delta}|_G \]

and because \(\alpha\) is regular, greater than \(\mu\) and greater than each \(|A_\delta, \overline{A_\delta}|_G|\).

Case 2. \(G\) is connected. Let \(\tilde{G}\) be the quotient graph of \(G\) modulo its \(\alpha\)-classes. \(\tilde{G}\) may have loops and multiple edges. It is clear that any well-ordering \(\leq_{\tilde{G}}\) on \(V(\tilde{G})\) such that each \(x \in V(\tilde{G})\) can be separated from \(\{y \in V(\tilde{G}) : y <_{\tilde{G}} x\}\) by a cut of cardinality less than \(\alpha\) of \(\tilde{G}\), when interpreted in \(G\), is a well-ordering with the required properties.

Since no two vertices of \(\tilde{G}\) are \(\alpha\)-edge-connected, and since \(\alpha^+\) is regular and uncountable, it follows by Corollary 3 that all blocks are of order \(\leq \alpha\). Let \(\Delta\) be the set of all blocks of \(\tilde{G}\). Note that \(\Delta\) is a decomposition of \(\tilde{G}\). Fix \(H_0 \in \Delta\) and define a partial order \(\leq_1\) on \(\Delta\) by:

\[ H \leq_1 K \iff H = K \text{ or } H = H_0 \text{ or every path of } \tilde{G} \text{ joining a vertex of } H_0 \text{ to a vertex of } K \text{ contains an edge of } H. \]

In other words, as in the proof of the Claim of Proposition 3, \(\leq_1\) is the partial order induced by the block-cutpoint tree of \(\tilde{G}\) rooted at \(H_0\). We leave it to the reader to show that, because of this tree structure, \(\leq_1\) can be refined to a well-ordering, say \(\leq_\Delta\).

For each \(x \in V(\tilde{G})\), denote by \(\delta(x)\) the \(\leq_\Delta\)-smallest element of \(\Delta\) that contains \(x\), and let \(\phi : \Delta \to \lambda\) be any injective function whose codomain is an ordinal and which satisfies \(H \leq_\Delta K \iff \phi(H) \leq \phi(K)\) for any \(H, K \in \Delta\). For any \(H \in \Delta \setminus \{H_0\}\), there exists a unique vertex \(x_H \in V(H)\) such that \(\delta(x_H) \neq H\). Let \(x_{H_0}\) be any vertex of \(V(H_0)\).

For each \(H \in \Delta\) define a well-ordering \(\leq_H\) on \(V(H)\) in which \(x_H\) is the smallest element and such that \((V(H), \leq_H)\) is embeddable into the well-ordering of \(\alpha\) as an ordinal, and let \(\psi_H : H \to \alpha\) be the order embedding. Finally define

\[ \theta : V(\tilde{G}) \to \lambda \times \alpha \]

\[ x \mapsto (\phi(\delta(x)), \psi_{\delta(x)}(x)), \]

where \(\lambda \times \alpha\) is the well-ordered set obtained by the lexicographic order on the cartesian product.

Clearly \(\theta\) is injective, and a well-ordering \(\leq_\Theta\) on \(V(\tilde{G})\) is defined by \(x \leq_\Theta y \iff \theta(x) \leq \theta(y)\). We will show that \(\leq_\Theta\) has the required separation property.

Because of the lexicographic structure which induces \(\leq_\Theta\) and because of the choice of the \(x_H\)'s, \(\leq_\Theta\) restricted to any \(V(H)\) coincides with \(\leq_H\). Since \(|H| \leq \alpha\) for any \(H\), by the claim of Case 1, \(\leq_H\) must have the property stated in the proposition.

Let \(x \in V(\tilde{G})\) and \(S := \{y \in V(\tilde{G}) : y <_{\tilde{G}} x\}\), and suppose by way of contradiction that all cuts separating \(x\) from \(S\) are of cardinality \(\geq \alpha\). The remark in the preceding paragraph implies that \(x \notin V(H_0)\) since otherwise \(S = \{y \in V(H_0) : y <_{H_0} x\}\).

Put \(K := \delta(x)\) and \(S_K := S \cap V(K)\). Observe that \(H_0 <_\Delta K\), and that \(y <_K x\) for any \(y \in S_K\). Moreover, \(x_K \in S_K\), since otherwise \(x = x_K\), which contradicts the fact that \(\delta(x_K) <_\Delta K = \delta(x)\). Thus, \(\leq_K\) having the required separation property, there

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exists a cut $C = [A, \overline{A}]_K$ of $K$ of cardinality $< \alpha$ such that $x \in V(A)$ and $S_K \subseteq V(\overline{A})$. If $A_x$ is the component of $A$ containing $x$, then $[A_x, \overline{A} x]_K \subseteq [A, \overline{A}]_K$; hence without loss of generality we may assume $A$ to be connected. $C$ must be non-empty because $K$ is connected and $S_K \neq \emptyset$, and moreover, since $K$ is a block of $G$, $C$ is also a cut of $G$. Since $G$ is connected, there is a unique induced subgraph $B$ of $G$ such that $A \subseteq B$, $\overline{A} \subseteq \overline{B}$ and $[B, \overline{B}]_G = C$. Moreover, since $A$ is connected, so also is $B$.

To finish the proof, let us show that $S \subseteq V(\overline{B})$. By way of contradiction, suppose there exists $z \in S \cap V(B)$. Being connected, $B$ must contain an $x_z$-path $P$. Since $S_K \subseteq V(\overline{A}) \subseteq V(\overline{B})$, $z$ cannot belong to $S_K$, and hence $z \notin V(K)$, i.e., $\delta(z) \neq K$. Moreover, $\delta(z) < K$ because of the lexicographic structure of $\leq_\Theta$. This implies that $P$ contains $x_K$, contradicting the fact that $P \subseteq B$ and $x_K \in S_K \subseteq V(\overline{A}) \subseteq V(\overline{B})$.

Case 3. $G$ is not connected. Left to the reader. \hfill \square

Note that Theorem 7 is not true for singular cardinals, not even for $\aleph_\omega$, the first one. The following is a counterexample.

Let $T$ be the dyadic tree rooted at $x_0$, and $\mathcal{R} := \{R_i\}_{i \in \mathbb{N}}$ the set of distinct $x_0$-rays of $T$. To construct the counterexample $G$, define a new vertex $u_R$ for each $R \in \mathcal{R}$, and then add $\aleph_\omega(V(R \cap R'))$ internally disjoint paths of length two connecting each pair of distinct vertices $u_R$, $u_R'$.

Let $u_{R_1}$ and $u_{R_2}$ be any two vertices of $G$, and $P$ the smallest initial segment of $R_1$ that is not an initial segment of $R_2$. Thus $P$ is the path $R_1 \cap R_2$ plus exactly one edge. Then, define $A_P$ as the subgraph of $G$ composed of all the $u_R$’s for which $P$ is an initial segment of the ray $R$, together with all the paths of length two that connects those $u_R$’s. Clearly $[A_P, \overline{A}_P]_G$ separates $u_{R_1}$ from $u_{R_2}$, and since $|\mathcal{R}| = \aleph_1$, it is easy to see that $|[A_P, \overline{A}_P]_G| = \aleph_{\omega - 1}$. This implies that no two vertices of $G$ are $\aleph_\omega$-edge-connected, and that if a well-ordering $\leq$ of $V(G)$ (viewed as the set of all the $\aleph_\omega$-classes of $G$ had the property of Theorem 7, then $u_{R} \cap R$ would be finite for each $x_0$-ray $R$ of $T$, where $u_R$ is the union of all $x_0$-rays $R'$ of $T$ for which $u_{R'} < u_R$. In consequence, each $x_0$-ray $R$ of $T$ would have a vertex $v_R \notin V(u_R)$ and $v_{R'}, v_R$ be distinct when $R \neq R'$ because either $R' \subseteq u_R$ or $R \subseteq u_{R'}$. This is impossible because $T$ has countably many vertices but uncountably many $x_0$-rays.

Any well-ordering of the set of $\alpha$-classes of $G$ can be extended to a well-ordering on $V(G)$, and it is easy to see that, if the well-ordering on the $\alpha$-classes has the property of Theorem 7, then any such extension has the compactness property stated in the following corollary. We shall extend the notion of $\alpha$-edge-connectivity between two vertices introduced in Section 2, by saying that a set $X \subseteq V(G)$ is $\alpha$-edge-connected to a vertex $x \in V(G)$ if there exist $\alpha$ edge-disjoint $x$-$X$-paths, or equivalently if $X$ cannot be separated from $u$ by a cut of $G$ of cardinality $< \alpha$.

**Corollary 5** Given a regular cardinal $\alpha$, the set of vertices of any graph $G$ can be well-ordered in such a way that for each pair $X \subseteq V(G)$, $u \in V(G)$ such that $u$ is an upper bound of $X$, the set $X$ is $\alpha$-edge-connected to $u$ if and only if some finite subset of $X$ is $\alpha$-edge-connected to $u$. (The finite subset can be chosen to be a singleton.)
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