Decompositions of infinite graphs: Part II
Circuit decompositions

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Abstract

We characterize the graphs that admit a decomposition into circuits, i.e. finite or infinite connected 2-regular graphs. Moreover, we show that, as is the case for the removal of a closed eulerian subgraph from a finite graph, removal of a non-dominated eulerian subgraph from a (finite or infinite) graph does not change its circuit-decomposability or circuit-indecomposability. For cycle-decomposable graphs, we show that in any end which contains at least $n + 1$ pairwise edge-disjoint rays, there are n edge-disjoint rays that can be removed from the graph without altering its cycle-decomposability. We also generalize the notion of the parity of the degree of a vertex to vertices of infinite degree, and in this way extend the well-known result that eulerian finite graphs are circuit-decomposable to graphs of arbitrary cardinality.

1 Introduction

As is well known, the subject of eulerian graphs originates from the Königsberg bridge problem that was solved in 1736 by Euler. In the finite case, the principal theorem on this topic (due in part to Euler, Hierholzer and Veblen), says in substance that for a connected graph, being edge-traceable (i.e., having a closed eulerian trail), having a decomposition into connected 2-regular subgraphs (circuits) and being eulerian are equivalent properties. In the infinite case, these three properties are no longer equivalent. It is known that for a finite or infinite graph, to be eulerian (i.e., not to contain any odd vertices) is equivalent to admitting a decomposition into edge-traceable graphs. Also, Nash-Williams has shown that the graphs which admit a decomposition into *finite* connected 2-regular subgraphs (cycles) are exactly the ones that contain no edge-cut of odd cardinality. The present paper characterizes the graphs that admit a decomposition into circuits (i.e. finite or infinite connected 2-regular graphs).

In contrast to the two characterizations just mentioned, no characterization of the graphs that are circuit-decomposable may be based only on the degrees of the vertices or on the cardinality of the edge-cuts. Indeed, there exist graphs G and H which differ by a single edge joining two vertices which cannot be separated by a finite cut in either, such that G is decomposable whereas H is not. (For example, take the graph of Figure 4, Section 4 to be H and the same graph minus the edge joining the two vertices of infinite degree for G.)

We approach the problem of the characterization in two ways.

Our first approach is to introduce a notion of local circuit-decomposability for certain specified parts of a graph, and then to show that if those specified parts of the graph are locally circuit-decomposable then the graph is circuit-decomposable in its entirety. The interesting parts of the graph in this approach are called *regions*: they are connected induced subgraphs that are joined to the rest of the graph by only a finite (even or odd) number of edges. Regions are said to be even or odd according as this number of edges is even or odd. Among all the regions, the *peripheral* ones, which are odd regions not containing any odd cut of the graph, play a key role in this approach since it turns out that a graph is circuit-decomposable if and only if its peripheral regions are locally circuit-decomposable (Theorem 7.5).

Hence the peripheral regions are the parts of the graph where one may expect to encounter serious difficulties in connection with circuit decomposition. At the other extreme, the regions that are most easily handled are those that are locally cycle-decomposable; by Nash-Williams's Theorem, such regions must be even and, like peripheral regions, cannot contain odd cuts. If one considers the two types of regions (peripheral and locally cycle-decomposable) in the case of finite graphs, one easily sees that they have the following two properties: (1) peripheral regions always contain at least one odd vertex; (2) locally cycle-decomposable regions contain even vertices only. For infinite graphs, this is no longer true because of the vertices of infinite degree that may be considered to be both of even and odd degree. This leads us to a generalization of the parity of the degree of a vertex to what we call the *parity type*. This is done in such a way that properties (1) and (2) extend to infinite graphs, and that the following conditions are satisfied:

(3) any two vertices in the same ω-class (i.e. which are connected by infinitely many pairwise edge-disjoint paths) have equal parity type;

(4) as for the parity of the degree, the removal from the graph of a finite eulerian subgraph does not alter the parity type of any vertex.

With this definition we show that having no *odd-type* vertex is a sufficient condition for a graph to admit a circuit decomposition, whereas having no *odd* vertex is a necessary condition. In particular, this gives the result that an infinite vertex in a transitive graph is always of even type, and hence for these graphs to be eulerian is a necessary and sufficient condition for the existence of a circuit decomposition.

Furthermore a peripheral region must always contain an odd number of ω-classes of odd-type vertices. This corresponds to the situation in the finite case where every peripheral region contains an odd number of odd vertices (note that in a finite graph the ω-classes are singletons). Moreover, if a peripheral region A of a graph G contains exactly one odd-type ω-class, then A is always locally cycle-decomposable *to within a single edge*, even if it is not locally circuit-decomposable. Hence, to study the circuit-decomposability of a graph, we can restrict ourselves to regions which are "almost" cycle-decomposable and therefore have a much simpler structure than the whole graph.

Odd-type and even-type vertices also play an important role in our second approach to the problem. This approach is based on the ray structure of the graph: we say that a graph G has *enough rays* if any odd region of G contains a ray. It is easy to see that having enough rays is a necessary condition for a graph to admit a circuit decomposition. On the other hand — relativizing the definition in the obvious way — we show that having enough non-dominated rays is a sufficient condition (Theorem 6.9); in fact, we show that it is a necessary and sufficient condition for a graph to admit a decomposition into *non-dominated* circuits. This leads us to ask whether having enough rays with some specified property might be a necessary and sufficient condition for a graph to admit a circuit decomposition. Indeed, such a property exists and will be called *eligibility*; to describe it we need the following observations.

It follows from Theorem 6.9 that, as for the eulerian subgraphs of a finite graph, the removal of an eulerian non-dominated subgraph H from a graph G does not affect its circuit-decomposability or circuit-indecomposability (see Theorem 8.1). Defining an *eulerian-type* graph as a graph that contains no odd-type vertex, we further show (Theorem 8.6) that this property also holds if H is a dominated eulerian-type subgraph provided that none of its dominating vertices is of odd type in the graph $G\backslash H$. In both situations we speak of *removable* subgraphs. In terms of this definition, we say that a ray R is *eligible* if it is contained in an "almost eulerian" locally finite removable graph. Any eligible ray of a circuit-decomposable graph is always contained in a 2-ray (a graph that is the edge-disjoint union of two rays having the same origin) that is a member of a decomposition of G into cycles and 2-rays. Since any decomposition into cycles and 2-rays trivially induces a circuit decomposition, we can see eligible rays as the rays that can be used if one want to construct a circuit decomposition. Moreover, our main theorem (Theorem 9.5) says that a graph G has a circuit decomposition if and only if it has enough eligible rays.

Eligible rays exist in most ends (i.e., classes of finitely inseparable rays). More precisely, eligible rays occur in every non-dominated end, in every end that contains at least three edge-disjoint rays (i.e. is of ϵ-multiplicity ≥ 3), and also in every end that is of ϵ-multiplicity ≥ 2 and is dominated by some odd-type vertex.

The preceding results show that a non-circuit-decomposable graph must contain an odd region whose ends are very "thin". Further, these graphs must contain an odd region which is of one of the four types that are shown in Figure 9 (Section 9) or a suitable combination of them.

2 Definitions and preliminaries

In this section we introduce basic definitions and present results that will be needed in the proofs leading up to our main theorem. Most of these results are known, but some are new. The latter have been grouped together in this section because they are also of independent interest.

2.1 Generalities

For the purposes of this paper, we assume all graphs to be unoriented, without loops or multiple edges, unless otherwise stated (multiple edges are allowed in quotient graphs). The symbol G will always denote a graph. An *eulerian* graph is a graph (not necessarily connected) whose vertices are all of even or infinite degree. *Paths* are understood to be finite. For $X, Y \subseteq V(G)$ an XY*-path* is a path whose endpoints are in X and Y, respectively; when X or Y is a singleton we omit curly brackets. If S, T are subgraphs of G, an ST*-path* is a $V(S)V(T)$-path. We shall speak of an X*-path* or an S*-path* in place of an XX-path or an SS-path respectively. A *ray* is a connected graph having exactly one vertex of degree one, called the *origin* of the ray, and all the others of degree 2. Given $X \subseteq V(G)$, an X*-ray* is a ray having its origin in X. As in the case of paths we shall also speak of u-rays and S-rays, where $u \in V(G)$ and S is a subgraph of G. A set of κ edge-disjoint rays $(R_i)_{i\in I}$, where κ is some cardinal, is said to be a κ*-tresse* if $\bigcap_{i\in I} V(R_i)$ is an infinite set.

A *circuit* is a non-empty connected 2-regular graph. A finite circuit is called a *cycle* and an infinite one, a *double-ray*. A *tail* of a ray or double-ray R is a subray of R. A *trail* is a sequence of consecutively adjacent vertices such that the edges joining two consecutive vertices of the sequence are all distinct. A trail may be finite, 1-way infinite or 2-way infinite. Whenever convenient the word trail will also be applied to the graph formed by the vertices and edges of the sequence. An *edge-traceable* graph is a graph obtained from a circuit by identifying some of its vertices but no edges and that contains no loop and no multiple edge. In other words, an edge-traceable graph is a graph which is either a finite eulerian trail or a 2-way infinite (and hence eulerian) trail. The finite edge-traceable graphs are therefore exactly the connected eulerian ones; for the infinite edge-traceable graphs, i.e., the ones that are a 2-way infinite trail, Erdős, Grünwald (Gallai) and Vázsonyi [3] give the following characterization:

Theorem 2.1 *A connected graph G is a 2-way infinite trail if and only if*

(i) $E(G)$ *is countably infinite;*

(ii) G *is eulerian;*

(iii) *there is no finite set of edges whose deletion leaves more than two infinite components; and*

(iv) *there is no finite eulerian subgraph the deletion of whose edges leaves more than one infinite component.*

Note that Erdős, Grünwald and Vázsonyi [3] also give a similar characterization of graphs that are a 1-way infinite trail (i.e., graphs obtained from a ray by identifying some of its vertices but no edges).

Given two subgraphs A, B of a graph G, we denote by $[A, B]_G$ the set of edges of G that have one endpoint in $V(A)$ and the other in $V(B)$, and $B - A$ denotes the induced subgraph of G on $V(B) - V(A)$. By abuse of language, we will frequently identify a single vertex with the graph that consists of this vertex only, and similarly a single edge may be identified with the graph that consists only of this edge and its two incident vertices. Thus in the preceding definitions, A can also be a vertex or a single edge. A *cut* of a graph G is a set of edges of the form $[A, G - A]_G$. If no confusion is likely, we shall write $\overline{A}$ instead of $G - A$, and, unless otherwise stated, both A and B will be *induced* subgraphs of G. A *region* of a graph G is a connected non-empty induced subgraph A such that $[A, \overline{A}]_G$ is finite (possibly empty); A is an *even* or *odd region* according as $|[A, \overline{A}]_G|$ is even or odd. A *bond* is a non-empty cut which is minimal with respect to inclusion. Observe that a cut $[A, \overline{A}]_G$ of a connected graph is a bond if and only if both A and $\overline{A}$ are connected. In general, every cut $[A, \overline{A}]_G$ is a union of edge-disjoint bonds (this is well known in the finite case; for a proof in the general case, see [11] Remark 1). Hence, in the case where $[A, \overline{A}]_G$ is finite, each component K of A is a region of G since $[K, \overline{K}]_G \subseteq [A, \overline{A}]_G$. The following lemma is useful in showing how one can construct finite cuts by transferring vertices from one side of a finite cut to the other.

Lemma 2.2 *Let $[A, \overline{A}]_G$ and $[B, \overline{B}]_G$ be two finite cuts of G. Then $[A-B, \overline{A - B}]_G$ and $[A \cap B, \overline{A \cap B}]_G$ are also finite cuts, and if, moreover, $[B, \overline{B}]_G$ is even and $B \subseteq A$, then $[A, \overline{A}]_G$ and $[A - B, \overline{A - B}]_G$ are either both even or both odd.*

Proof The first assertion follows from the fact that both $[A - B, \overline{A - B}]_G$ and $[A \cap B, \overline{A \cap B}]_G$ are contained in $[A, \overline{A}]_G \cup [B, \overline{B}]_G$. For the second assertion, note that when $B \subseteq A$, we have

$$\begin{aligned} |[A - B, \overline{A - B}]_G| &= |[A, \overline{A}]_G| - |[B, \overline{A}]_G| + |[B, A - B]_G| \\ &= |[A, \overline{A}]_G| - |[B, \overline{A}]_G| + |[B, \overline{B}]_G| - |[B, \overline{A}]_G| \\ &= |[A, \overline{A}]_G| - 2|[B, \overline{A}]_G| + |[B, \overline{B}]_G|. \end{aligned}$$

□

If A is a set of edges, $G \backslash A$ is the graph obtained by the removal of all edges of A (retaining all vertices). For a non-empty graph H, $G \backslash H$ denotes the graph $G \backslash E(H)$. $\mathrm{Bdry}_G(H)$ is the set of all vertices of H that are incident with an edge of $G \backslash H$. G/H is the quotient graph obtained by identifying all the vertices of H and by removing all the loops thus obtained; G/H may have multiple edges. We extend the definition of a quotient graph A/H to the case where A is a subgraph of G by taking $A/H := A/(A \cap H)$. There will be situations in which the set $V(H)$ appears both as a set of vertices of G and as a vertex of G/H; to avoid confusion we shall denote $V(H)$ by q_H when it is a vertex of the quotient.

Two vertices x and y are said to be *infinitely edge-connected* if there exist infinitely many pairwise edge-disjoint xy-paths or equivalently if the two vertices cannot be separated by a finite cut of G. This is an equivalence relation on $V(G)$; its classes are called *ω-classes*.

The next lemma shows that it is possible to transfer an ω-class (possibly with some other vertices) from one side of a finite cut to the other and preserve the finiteness of the cut.

Lemma 2.3 *Let A be an induced subgraph of G such that $[A, \overline{A}]_G$ is finite, $x \in V(A)$ and Y be a finite set of vertices that is disjoint from the ω-class of x. Then there exists a region $B \subseteq A$ that contains x (and hence its ω-class) but no vertex of Y.*

Proof For each $y \in Y$, fix a finite cut $[B_y, \overline{B_y}]_G$ such that $y \in V(B_y)$ and $x \in V(\overline{B_y})$ and put $C := A \cap \bigcap_{y \in Y} \overline{B_y}$. Then $[C, \overline{C}]_G$ is finite since it is contained in $[A, \overline{A}]_G \cup \bigcup_{y \in Y} [B_y, \overline{B_y}]_G$. Therefore the component of C that contains x is the desired region. □

2.2 Dominated rays and end-equivalent rays

A ray R is said to be *dominated* (resp. *ϵ-dominated*) by a vertex x in a graph G if for any finite set $S \subseteq V(G) - \{x\}$ (resp. $S \subseteq E(G)$), some tail of R lies in the same component of $G - S$ (resp. $G \backslash S$) as x or, equivalently, if there exist infinitely many xR-paths of G which pairwise intersect in x only (resp. which are pairwise edge-disjoint and have different endpoints in R). By abuse of language we will frequently omit mention of the dominating vertex and simply speak of dominated rays. If H is a subgraph of G the statement "a ray R is dominated in H" will mean that R is contained in H, and is dominated in H by a vertex of H. The same convention will be applied to ϵ-domination. Note that a ray which is dominated (resp. ϵ-dominated) in H is still dominated (resp. ϵ-dominated) in G, whereas the converse is not necessarily true. However, if H is a region, then it is easy to see that a ray of H is dominated (resp. ϵ-dominated) in H if and only if it is dominated (resp. ϵ-dominated) in G. For this reason, we shall not state explicitly whether a ray of a region is dominated (resp. ϵ-dominated) in G or in the region. It is obvious that any vertex which dominates some ray in a graph also ϵ-dominates it; however, as is shown in Proposition 2.19 and by Figure 1, the converse is not true. Further, the set of all the vertices which ϵ-dominate some ray is always an ω-class of the graph, provided it is non-empty; but this is not true of the vertices which dominate a ray.

Two rays R_1, R_2 are *end-equivalent* in a graph G (in symbols, $R_1 \sim R_2$) if for every finite subset S of $V(G)$, some tails of R_1 and R_2 lie in the same component of $G - S$ or, equivalently, if there exist infinitely many (vertex-)disjoint R_1R_2-paths of G. It is well known that $\sim$ is indeed an equivalence relation. The equivalence classes are called *ends*. (See Diestel [1] for a survey.) Clearly, a κ-tresse is always contained in a single end. Given a set of ends $\mathcal{R}$, we will abuse language and say that a ray belongs to $\mathcal{R}$ if it belongs to an end of $\mathcal{R}$. An end or a κ-tresse τ is said to be *dominated* (resp. *ϵ-dominated*) by a vertex x in G if x dominates (resp. ϵ-dominates) some (and therefore all) rays in τ; again we will frequently omit mention of the dominating vertex. The *ϵ-multiplicity* of an end τ is the maximal number of edge-disjoint rays of τ, or to be more precise:

$$\textit{ϵ-multiplicity} \text{ of } \tau := \sup\{|\rho| : \rho \text{ is a set of edge-disjoint elements of } \tau\}.$$

As in the case of the multiplicity defined by Halin [8] (supremum of the number of vertex-disjoint rays in τ), the supremum for the number of edge-disjoint rays is actually attained. The proof is similar to that of Halin's [8] Satz 1, which is itself relies on Halin's [7] Satz 1.

Remark 2.4 An end τ that has infinitely many dominating vertices always contains an ω-tresse, and hence has infinite ϵ-multiplicity. One can easily construct an ω-tresse in τ in the following way. Let D be the set of vertices that dominate τ in G, and let $x_0 \in D$. Let P_0^0 be any non-degenerate x_0D-path of G, and x_1 be the end-vertex of P_0^0 that is not x_0. Let P_0^1 be any non-degenerate x_1D-path of $G-(P_0^0-\{x_1\})$, x_2 be the end-vertex of P_0^1 that is not x_1, and P_1^0 be any x_0x_1-path of $G-(P_0^0 \cup P_0^1 - \{x_0, x_1\})$. Let P_0^2 be any non-degenerate x_2D-path of $G-(P_0^0 \cup P_0^1 \cup P_1^0 - \{x_2\})$, x_3 be the end-vertex of P_0^2 that is not x_2, P_1^1 be any x_1x_2-path of $G-(P_0^0 \cup P_0^1 \cup P_1^0 \cup P_0^2 - \{x_1, x_2\})$, and P_2^0 be any x_0x_1-path of $G-(P_0^0 \cup P_0^1 \cup P_1^0 \cup P_0^2 \cup P_1^1 - \{x_0, x_1\})$. Continue this construction, and put $R_n := \bigcup_{i\in\omega} P_n^i$ to obtain the desired family.

One of the best known results on ends is Halin's theorem [6] on the existence of an end-faithful spanning tree for countable connected graphs.

Theorem 2.5 *Every connected countable graph G contains an end-faithful spanning tree, i.e., a spanning tree T such that*

(1) *no two disjoint rays of T are end-equivalent in G;*

(2) *every ray of G is end-equivalent in G to some ray of T*

A particularly interesting case of end-faithful spanning tree is the normal tree.

Definition 2.6 A spanning tree T with root r of a graph G is called *normal* if the endvertices of every edge of G are comparable in the natural tree order on $V(G)$ induced by T.

Normal spanning trees are known in the finite case as depth-first search trees, and it is a direct consequence of the following lemma that they always are end-faithful.

Lemma 2.7 (Diestel and Leader [2]) *Let T be a normal tree of a graph G. Then every ray in G meets some ray of T infinitely often.*

The next theorem characterizes the graphs that admit normal trees. Since a finite set of vertices is always a dispersed set (see below for the definition), it follows directly from that theorem that normal trees always exist in countable graphs.

Theorem 2.8 (Jung [9]) *A connected graph G has a normal spanning tree if and only if $V(G)$ is a countable union of dispersed sets, where a dispersed set is a set of vertices such that every ray is separable from it by a finite number of vertices of G.*

The last two results imply the following corollary that we will need latter on.

Corollary 2.9 *Any end τ of a countable graph contains a ray that meets any other ray of τ infinitely often.*

An end-faithful spanning tree is often viewed as a representative of the ends of the original graph. The next result is about spanning trees that represent no end at all.

Theorem 2.10 (Polat [14] and Širáň [16]) *A connected countable graph has a rayless spanning tree if and only if every ray in G is dominated.*

The next three propositions show how domination and end-equivalence are linked.

Proposition 2.11 [12] (or [5] for a more direct proof) *Given an infinite set of vertices X of a connected graph G, there exist a vertex $x \in V(G)$ and an infinite family of xX-paths which pairwise intersect in x only, or a ray Q and an infinite family of vertex-disjoint QX-paths.*

For our purposes, the following consequence of this proposition is more relevant.

Proposition 2.12 *Let R be any ray of G. If a connected subgraph A of $G \backslash R$ contains infinitely many vertices of R, then there exists a vertex $x \in V(A)$ that dominates R in $A \cup R$, or a ray $Q \subseteq A$ that is end-equivalent to R in $A \cup R$.*

Proposition 2.13 *Let R, R' be two end-inequivalent rays of G, and $x \in V(G)$. If x dominates R in G then x will dominate a tail of R in both $G - (R' - x)$ and $G \backslash R'$.*

Proof Since $G - (R' - x)$ is a subgraph of $G \backslash R'$, we only have to show the result for $G - (R' - x)$. Let $\mathcal{F}$ be an infinite set of xR-paths of G which pairwise intersects in x only. If x does not dominate R in $G - (R' - x)$, then $R' - x$ must meet all but a finite number of paths of $\mathcal{F}$. Since each such path of $\mathcal{F}$ contains an RR'-path disjoint from x, R and R' are therefore end-equivalent in G, a contradiction. $\square$

Definition 2.14 A family $(H_n)_{n \in \omega}$ of connected subgraphs of G is said to be a *stratifying sequence* if $\mathrm{Bdry}(H_n)$ is a non-empty finite set and $H_{n+1} \subseteq H_n - \mathrm{Bdry}(H_n)$ for any n. If, moreover, $[H_n, \overline{H_n}]_G$ is finite for any n (i.e., if each H_n is a region), we then speak of an ϵ-*stratifying sequence*.

Remark 2.15 Given a stratifying sequence $(H_n)_{n \in \omega}$, it follows from the second property of the definition that a vertex in H_n must be at distance at least n from any vertex of $\mathrm{Bdry}(H_0)$. Therefore $\bigcap_{n \in \omega} H_n = \emptyset$.

Remark 2.16 If a ray R meets infinitely many members of a stratifying sequence then each member of that sequence contains a tail of R.

The next proposition shows that there is a strong link between stratifying sequences and non-dominated ends.

Proposition 2.17

(i) *A ray of a graph G is not dominated (resp. not ϵ-dominated) in G if and only if it meets every member of some stratifying (resp. ϵ-stratifying) sequence;*

(ii) *the set of all the rays that meet each member of a stratifying (resp. ϵ-stratifying) sequence is a non-dominated (resp. non-ϵ-dominated) end.*

Proof We shall give a proof for the dominated case; the ϵ-dominated case is very similar and left to the reader.

(i)–*Necessity.* Suppose R is not dominated in G and observe that for any finite set of vertices S, there exists another finite set of vertices $X \subseteq V(G) - S$ such that the unique component of $G - X$ that contains a tail of R does not meet S. Otherwise, by Menger's Theorem, there exists an infinite family of SR-paths any two of which are either disjoint or meet on their end-vertex in S only. Hence some $s \in S$ must be an end-vertex of an infinite number of those SR-paths, contradicting the hypothesis that R is not dominated.

We will now construct two sequences $(X_n)_{n\in\omega}$ and $(H_n)_{n\in\omega}$, consisting of finite subsets of $V(G)$ and connected subgraphs of G, respectively. Let x be any vertex belonging to the component of G that contains R, and X_0 be a finite set of vertices of $V(G) - x$ such that the unique component of $G - X_0$ that contains a tail of R does not contain x. Let H_0 be that component. Then let X_1 be a finite set of vertices of $V(G) - (X_0 \cup \{x\})$ such that the component of $G - X_1$ that contains a tail of R does not meet $X_0 \cup \{x\}$. Let H_1 be that component. Repeat this construction mutatis mutandis to obtain the sequence $(H_n)_{n\in\omega}$. Clearly the sequence is stratifying since each H_n has its boundary contained in the finite set X_n.

Sufficiency. By Remark 2.16, each H_n contains a tail of R. $\mathrm{Bdry}(H_n)$ being finite, any vertex that dominates R in G must belong to H_n, for all n, a contradiction to Remark 2.15.

(ii)–Let $(H_n)_{n\in\omega}$ be a stratifying sequence of G and τ the set of all rays that meet each H_n. Since $X := \bigcup_{n\in\omega} \mathrm{Bdry}(H_n)$ is an infinite set of vertices, it follows from Proposition 2.11 that there exist a vertex $x \in V(G)$ and an infinite family of xX-paths which pairwise intersect in x only, or a ray Q and an infinite family of vertex-disjoint QX-paths. In this case the first possibility can not occur because such an x has to belong to infinitely many H_n's, and therefore to all of them, contradicting Remark 2.15. Since any such ray Q must clearly belong to τ, the latter is therefore non-empty, and it is then easy to see that τ is an end. Moreover, τ is a non-dominated end since, by (i), no ray of τ is dominated. $\square$

Corollary 2.18 *Let H be a connected subgraph of G all of whose rays are dominated (resp. ϵ-dominated) in G. Then, for any stratifying (resp. ϵ-stratifying) sequence $(H_n)_{n\in\omega}$ of G, there exists $n_0 \in \omega$ such that $V(H) \cap V(H_n) = \emptyset$ for any $n \geq n_0$.*

Proof The H_n's being stratifying, clearly if $V(H) \cap V(H_n)$ is empty for some $n = n_0$, then it is also empty for any $n \geq n_0$. Hence, by way of contradiction suppose that $V(H) \cap V(H_n) \neq \emptyset$ for all n. Let T be a spanning tree of H and $x \in V(T)$ $(= V(H))$. By Remark 2.15, there is an index n_1 such that $x \notin V(H_n)$ for any $n \geq n_1$, and let T_1 be the union of all xy-paths of T with $y \in \bigcup_{n\geq n_1} \mathrm{Bdry}_G(H_n)$. Clearly T_1 is infinite but since T_1 is a subtree of T and the H_n's are stratifying, $T_1 - H_n$ is finite for any $n \geq n_1$. Thus T_1 is an infinite locally finite connected graph, and therefore contains an x-ray R (say). It follows from the finiteness of each $T_1 - H_n$ that R meets each H_n for any $n \geq n_1$ (and hence for any n). Thus, by Proposition 2.17 (i), R is not dominated (resp. ϵ-dominated) in G, contrary to hypothesis. □

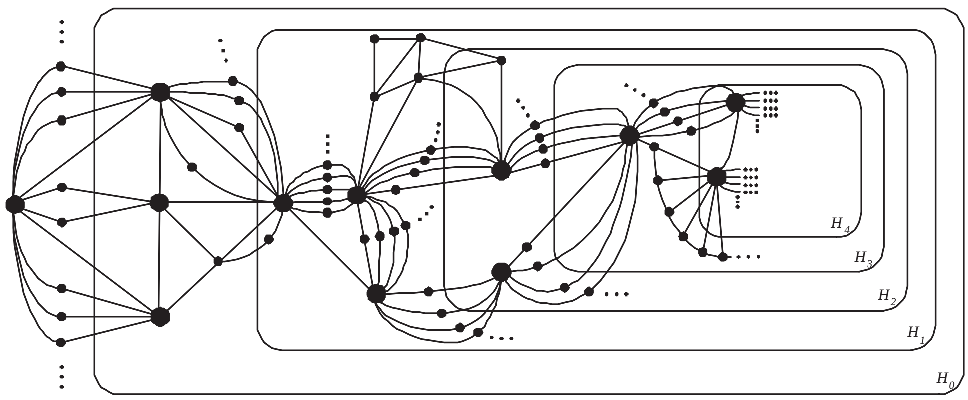

Figure 1: A graph having a non-dominated ϵ-dominated end.
The rays contained in $H_3 \backslash H_4$ do not belong to that end.

The next proposition says that an end that is ϵ-dominated but not dominated must be very similar in structure to the unique end of the graph in Figure 1.

Proposition 2.19 *Let τ be a set of rays of G and U an ω-class of G. Then the following statements are equivalent:*

(i) *τ is a non-dominated end that is ϵ-dominated by some (or equivalently, each) vertex of U;*

(ii) *There exists a stratifying sequence $(H_n)_{n\in\omega}$ such that U meets each H_n, and τ is the set of all rays that have a tail in each H_n;*

(iii) *τ is a non-dominated end that contains an ω-tresse $(R_i)_{i\in\omega}$ such that $\bigcap_{i\in\omega} V(R_i) \subseteq U$.*

Proof Without loss of generality, we may suppose G to be connected, because otherwise we may consider only the component of G that contains the ω-class U.

(i)⇒(ii). Let $R \in \tau$ and let $(H_n)_{n\in\omega}$ be a stratifying sequence given by Proposition 2.17 (i). By Remark 2.16, R has a tail in each H_n, and clearly a ray is end-equivalent to R in G if and only if it has the same property. Hence τ is the set of all rays that have a tail in each H_n. Now, by way of contradiction, suppose that $V(H_n) \cap U = \emptyset$ for some $n \in \omega$. Let R_n be a tail of R contained in H_n and u be an element of U. Since u ϵ-dominates R, there exist infinitely many edge-disjoint uR_n-paths. Since $u \in U \subseteq V(G) - V(H_n)$, each of these paths includes an element of Bdry(H_n). Since Bdry(H_n) is finite, it follows that u is connected to some $x \in \mathrm{Bdry}(H_n)$ by infinitely many edge-disjoint paths, and so $x \in U$, contradicting the assumption that $V(H_n) \cap U = \emptyset$.

(ii)⇒(iii). It follows from Proposition 2.17 (ii) that τ is a non-dominated end. Let $u \in U$. By Remark 2.15, there is an index n_0 such that $u \notin V(H_n)$ for all $n \geq n_0$. By truncating the sequence $(H_n)_{n\in\omega}$, we may therefore suppose that $u \notin V(H_0)$.

Claim: There exists an $X \subseteq \bigcup_{n\in\omega} \mathrm{Bdry}(H_n)$, and a connected subgraph T of G such that

(1) X meets each Bdry(H_n) (and hence is infinite);

(2) T contains u and X, and any $x \in X$ is infinitely edge-connected to u in T;

(3) given any two distinct vertices x, x' of X, either x separates x' from u in T or vice versa.

If we suppose the claim to be true then by (3), X is a set of cut-vertices of T, and moreover, X is strung out along a ray in the block-cutpoint tree of T. Hence, by (2), T contains an infinite family of edge-disjoint u-rays R_i, $i \in \omega$, each containing X. Clearly the R_i's form an ω-tresse, and by (1) each R_i belongs to τ. Moreover, $\bigcap_{i\in\omega} V(R_i)$ being contained in the ω-class of T containing u, it follows that $\bigcap_{i\in\omega} V(R_i) \subseteq U$ (the ω-class of G containing u), and we are done.

Proof of the claim Let $R \in \tau$. Since U is an ω-class that meets each H_n, and since the H_n's are connected, u ϵ-dominates R in G.

Choose an enumeration $\{x_1, x_2, \ldots\}$ of $\bigcup_{n\in\omega} \mathrm{Bdry}(H_n)$. Starting with $G_0 = G$, we inductively define a decreasing sequence of subgraphs G_i of G such that

(a) $x_{i+1} \in V(G_i)$, and G_{i+1} is defined from G_i either by the deletion of at most one vertex (viz. x_{i+1}) or the deletion of a finite number of edges (hence each G_i contains a tail of R);

(b) all tails of R contained in G_i are ϵ-dominated in G_i by u;

(c) if $x_i \in V(G_i)$ then it is a cut-vertex of G_i that separates u in G_i from some tail of R.

Suppose G_i has already been defined. In view of (c), we make the following case distinction:
Case 1. x_{i+1} is a cut-vertex of G_i which separates u from some tail of R. In this case put $G_{i+1} := G_i$.

Case 2. Some component K of $G_i - x_{i+1}$ contains both u and a tail of R. If u ϵ-dominates some tail of R in K, take $G_{i+1} := G_i - x_{i+1}$. If u does not ϵ-dominate any tail of R in K, then there exists a finite bond $[A, \overline{A}]_K$ that separates u from some tail of R. Put $G_{i+1} := G_i \backslash [A, \overline{A}]_K$.

It is easy to see that the G_i's satisfy conditions (a), (b) and (c), and that at most a finite number of H_n's are not subgraphs of G_i for any i.

Let T be the component of $\bigcap_{i \in \omega} G_i$ that contains u and $X := \{x_i : x_i \in V(T)\}$, and let us show they have the claimed properties. It is easy to see that $x_{i+1} \notin X$ if and only if $G_{i+1} = G_i - x_{i+1}$. This implies that Condition (1) is satisfied for any n because otherwise $G_{i+1} = G_i - x_{i+1}$ for every $x_{i+1} \in \mathrm{Bdry}_G(H_n)$, implying that there exists a finite number j such that $\mathrm{Bdry}_G(H_n) \cap V(G_j) = \emptyset$, contradicting (b) for $i = j$.

It follows from (b) and (c) that any vertex $x_i \in X$ is infinitely edge-connected to u in G_i, and hence, again because of (b), in each G_j with $j \geq i$. Moreover it follows from (c) that x_i separates u from some H_m. If i_m is the smallest index such that $x_j \in V(H_m)$ for any $j \geq i_m$, then every ux_i-path of G_{i_m} is contained in T. Condition (2) is therefore satisfied.

Let x_k, x_l $(k < l)$ be any two vertices of X. Since $G_l \subseteq G_k$ and because of (b) and (c), in the graph G_l, either x_k separates x_l from u or x_l separates x_k from u. Thus Condition (3) is satisfied because $T \subseteq G_l$ and because it follows from Condition (2) that u, x_k and x_l all belong to T.

(iii)⇒(i). Clearly τ is ϵ-dominated by each vertex of the infinite subset $\bigcap_{i \in \omega} V(R_i)$ of U. Hence τ is ϵ-dominated by each vertex of U. □

Corollary 2.20 *A non-dominated ϵ-dominated end has infinite ϵ-multiplicity.*

Corollary 2.21 *An ϵ-dominated end that contains no ω-tresse has a finite and strictly positive number of dominating vertices.*

Proof Remark 2.4 and Proposition 2.19. □

Among all the rays that are ϵ-dominated by a given vertex u, some seem "closer" to u than others; a closer ray being "in the middle" between u and the ray farther away. Formally, given two edge-disjoint rays Q and Q' that are ϵ-dominated by u, we say that Q *is closer to u than* Q' in G (denoted by $Q \prec_u Q'$) if Q' is not ϵ-dominated by u in $G \backslash Q$.

It follows from the definition of ϵ-domination that $Q \prec_u Q'$ if and only if given any infinite family $\mathcal{F}$ of edge-disjoint uQ'-paths having distinct end-vertices on Q', at most finitely many members of $\mathcal{F}$ are edge-disjoint from Q. Thus, it is easy to see that $\prec_u$ is a transitive relation. Moreover, since the paths of $\mathcal{F}$ can be chosen to be edge-disjoint from Q', $Q \prec_u Q'$ implies that $Q' \not\prec_u Q$. Hence $\preceq_u$ is a partial order. The following result shows that, under very mild conditions, $\prec_u$ has a certain Noetherian property.

Proposition 2.22 *Let u be a vertex that does not ϵ-dominate any ω-tresse in G. Then, for any infinite ascending chain $Q_0 \prec_u Q_1 \prec_u Q_2 \prec_u \ldots$ of edge-disjoint rays ϵ-dominated by u, all but a finite number of the Q_i's are end-equivalent.*

The proof is based on the following lemma.

Lemma 2.23 *A vertex u ϵ-dominates an ω-tresse in G if and only if there exists an infinite subset U of the ω-class of u such that for each finite set $S \subseteq V(G)$, all but a finite number of vertices of U belong to the same ω-class of $G - S$.*

Proof With $U := \bigcap_{i\in\omega} V(R_i)$ for some ω-tresse $(R_i)_{i\in\omega}$ ϵ-dominated by u, the necessity is an immediate consequence of the definition of ϵ-domination. For the sufficiency, for each finite set $S \subseteq V(G)$, denote by U_S the only infinite set which is of the form $U \cap X$ for some ω-class X of $G - S$, and let us recursively construct a ray that is ϵ-dominated by u in G as follows. Fix $x_0 \in U$. Let W_1 be an $x_0U_{\{x_0\}}$-path of G which is internally disjoint from $U_{\{x_0\}}$, and denote by x_1 the end-vertex of W_1 that belongs to $U_{\{x_0\}}$. It follows from $V(W_1) \cap U_{\{x_0\}} = \{x_1\}$, that $U_{\{x_0\}} = U_{V(W_1-x_1)}$. Hence $x_1 \in U_{V(W_1-x_1)}$, implying that x_1 and $U_{V(W_1)}$ lie in the same component of $G-(W_1-x_1)$. Let W_2 be an $x_1U_{V(W_1)}$-path of $G-(W_1-x_1)$ that is internally disjoint from $U_{V(W_1)}$, and denote by x_2 its end-vertex contained in $U_{V(W_1)}$. Since $x_2 \in U_{V(W_1\cup W_2-x_2)}$, we can then choose an $x_2U_{V(W_1\cup W_2)}$-path W_3 that is internally disjoint from $U_{V(W_1\cup W_2)}$, etc.

Clearly, $R := W_1 \cup W_2 \cup \cdots$ is a ray. Moreover, since R meets U (and hence the ω-class of u) infinitely often, R is ϵ-dominated by u in G.

Let τ be the end of R in G, and D its set of dominating vertices. If D is infinite then, by Remark 2.4, we are done. Hence suppose D to be finite (possibly empty). Then by Proposition 2.19 (i) $\Rightarrow$ (iii) applied to $G-D$, we also are done because a tail of R contained in $G-D$ is non-dominated in $G-D$ but ϵ-dominated by each vertex of U_D, and because any ray end-equivalent to a tail of R in $G-D$ is end-equivalent to R in G. □

Proof of Proposition 2.22 We first note that if Q_j and Q_k $(j < k)$ belong to the same end τ of G, then so do all Q_l's with $j \le l \le k$. Assume the contrary and choose l such that $j < l < k$ and $Q_l \notin \tau$. Since $Q_l \not\prec_u Q_j$, there exists an infinite family $\mathcal{F}_1$ of edge-disjoint uQ_j-paths, edge-disjoint from Q_l and pairwisely having different end-vertices on Q_j. Moreover, since $Q_l \notin \tau$, there also exists an infinite family $\mathcal{F}_2$ of vertex-disjoint Q_jQ_k-paths that are vertex-disjoint from Q_l. Since Q_l is edge-disjoint from Q_j, one can therefore construct an infinite family of edge-disjoint uQ_k-paths contained in $\bigcup\mathcal{F}_1 \cup Q_j \cup \bigcup\mathcal{F}_2$. This is a contradiction to $Q_l \prec_u Q_k$ because Q_l is edge-disjoint from $\bigcup\mathcal{F}_1 \cup Q_j \cup \bigcup\mathcal{F}_2$.

Suppose that the conclusion of the proposition is false. By the preceding paragraph, no end of G may contain an infinite number of Q_i's. In other words, there exists an infinite subsequence of the Q_i's that are pairwise end-inequivalent, and without loss of generality, this subsequence may be taken as the whole sequence itself.

Now, let us show that if Q_j and Q_k ($j < k$) are both dominated by some vertex x then $k = j + 1$. By way of contradiction, suppose $k > j + 1$. By Proposition 2.13, it follows from the end-inequivalence of the Q_i's that a tail of Q_j and a tail of Q_k are both dominated by x in $G \backslash Q_{j+1}$. Since moreover u still ϵ-dominates Q_j in $G \backslash Q_{j+1}$, we therefore have that for any finite set $S \subseteq E(G \backslash Q_{j+1})$, the vertex u, a tail of Q_j, the vertex x and a tail of Q_k lie all in the same component of $(G \backslash Q_{j+1}) \backslash S$. Thus u still ϵ-dominates Q_k in $G \backslash Q_{j+1}$, contradicting the fact that $Q_{j+1} \prec_u Q_k$.

Hence there is an infinite subsequence of the Q_i's no two of whose members share a dominating vertex; without loss of generality, suppose that it is the whole sequence.

Each Q_i has a dominating vertex x_i. This follows from the fact that Q_i is ϵ-dominated by u, and hence if Q_i were non-dominated then by Proposition 2.19, the end of Q_i would contain an ω-tresse, contradicting the hypothesis. The set $U := \{x_i : i \in \omega\}$ is infinite and contained in the ω-class of u. Hence by Lemma 2.23 there exists a finite set $S \subseteq V(G)$ such that for any i, $\{x_j : j \geq i\}$ meets more than one ω-class of $G - S$. Suppose that among all such sets, S has minimal cardinality. Since $S \neq \emptyset$, fix $s \in S$ and let i_0 be the smallest index such that $\{x_j : j \geq i_0\}$ is contained in a single ω-class of $G - (S - \{s\})$, and such that no Q_j ($j \geq i_0$) is dominated by s in G. Since each x_j dominates Q_j, $s \notin \{x_j : j \geq i_0\}$.

Let i_1 and i_2 be two indices with $i_0 < i_1 < i_2$ such that x_{i_1}, as a vertex of $G - S$, is neither in the ω-class of x_{i_0} nor in the one of x_{i_2} (x_{i_0} may be in the same ω-class as x_{i_2}).

We claim that there exists an infinite family $\mathcal{F}$ of edge-disjoint $x_{i_0}x_{i_2}$-paths of $G - (S - \{s\})$ that are edge-disjoint from Q_{i_1}. If x_{i_0} and x_{i_2} belong to the same ω-class of $G - S$, then the claim follows from the fact that x_{i_1} dominates a tail of Q_{i_1} in $G - S$. In that case, the paths of the desired family can even be chosen in $G - S$. If x_{i_0} and x_{i_2} belong to different ω-classes of $G - S$, then again because of the dominating property of x_{i_1} and because s does not dominate Q_{i_1} in G, there exists in $G - (S - \{s\})$ an infinite family of edge-disjoint $x_{i_0}s$-paths, and an infinite family of edge-disjoint sx_{i_2}-paths, which both consist of paths that are edge-disjoint from Q_{i_1}. Clearly, in the union of all the paths in these two families, one can construct the desired family $\mathcal{F}$.

To finish the proof, we will now construct an infinite family of edge-disjoint uQ_{i_2}-paths which are edge-disjoint from Q_{i_1} and have distinct end-vertices on Q_{i_2}; the existence of such a family being in contradiction with $Q_{i_1} \prec_u Q_{i_2}$. Let $\mathcal{F}_{i_k}$ ($k = 0$ or 2) be an infinite family of $x_{i_k}Q_{i_k}$-paths, which pairwise intersect in x_{i_k} only and are edge-disjoint from Q_{i_1}. Such families exist because x_{i_k} dominates Q_{i_k} in G, and Q_{i_k} and Q_{i_1} are end-inequivalent, $k = 0, 2$. Finally, since $Q_{i_0} \prec_u Q_{i_1}$, one can construct an infinite family $\mathcal{F}_u$ of edge-disjoint uQ_{i_0}-paths, pairwise having distinct end-vertices on Q_{i_0}, and all being edge-disjoint from Q_{i_1}.

It is easy to see that in $H := \bigcup \mathcal{F}_u \cup \bigcup \mathcal{F}_{i_0} \cup \bigcup \mathcal{F} \cup \bigcup \mathcal{F}_{i_2}$, there exists an infinite family of edge-disjoint uQ_{i_2}-paths with distinct end-vertices on Q_{i_2}, and we are done because $E(H) \cap E(Q_{i_1}) = \emptyset$. $\square$

2.3 Decompositions

A *decomposition* of a graph G is an equivalence relation on $E(G)$ such that the subgraph induced by the edges of any equivalence class is connected. The (edge-)induced subgraphs obtained in this way are called the *fragments* of the decomposition. Thus, a decomposition of a graph G may be considered as a family of edge-disjoint connected subgraphs of G whose union is the graph G minus its isolated vertices. Given a decomposition Δ of a graph G and a subgraph H, the Δ-*shadow of* H is the subgraph of G which is the union of all the fragments of Δ that edge-intersect H. An induced subgraph A of a graph G is said to be *locally cycle-decomposable* (resp. *locally circuit-decomposable*) in G if $G\backslash\overline{A}$ has a decomposition into cycles and $\overline{A}$-paths (resp. circuits, $\overline{A}$-rays and $\overline{A}$-paths) or, which is equivalent, if $G/\overline{A}$ is cycle-decomposable (resp. decomposable into circuits and $q_{\overline{A}}$-rays).

The decompositions relevant to our purposes are cycle decompositions (i.e. decompositions whose fragments are cycles), circuit decompositions and decompositions into edge-traceable graphs. Note that these three types of decompositions are closely related. Indeed it is a consequence of two classical results (Theorems 2.24 and 2.25 below) that the cycle-decomposable graphs are exactly the graphs having a decomposition into finite edge-traceable graphs, and that the circuit-decomposable graphs are those that have a decomposition into locally finite edge-traceable graphs.

In the finite case we have Veblen's Theorem that a finite graph has a cycle decomposition if and only if it is eulerian. This immediately generalizes to the locally finite case:

Theorem 2.24 *A locally finite graph has a circuit decomposition if and only if it is eulerian.*

For arbitrary graphs we have:

Theorem 2.25 **[13]***A graph has a decomposition into edge-traceable graphs if and only if it is eulerian.*

Recall that a *transition system* on a graph G is a family $\theta = (\theta_x)_{x\in V(G)}$ such that each θ_x is a partition into pairs of the set of edges incident with x (see [4, III.40]). From any transition system θ, one can construct the equivalence relation on $E(G)$ which is the transitive closure of the relation given by all the pairs of all the θ_x's. Any class of that equivalence relation induce then an edge-traceable graph, and the preceding theorem can therefore be stated in the following stronger way:

Theorem 2.26 **[15]***Any transition system on an eulerian graph G induces a decomposition of G into edge-traceable graphs.*

Graphs admitting a cycle decomposition have been characterized by Nash-Williams. This result – which we will refer to as Nash-Williams's theorem – can be formulated as follows:

Theorem 2.27 **[13, 15]***For any graph G, the following are equivalent:*

(1) *G has a cycle decomposition;*

(2) *G has no odd cut;*

(3) *every finite subgraph of G is contained in a finite eulerian subgraph of G.*

Since for any odd cut $[A, \overline{A}]_G$, A is always a disjoint union of regions, of which at least one is odd, condition (2) can be replaced by

($2'$) *G has no odd region.*

Proposition 2.28 *If a graph G has a decomposition Δ into circuits and rays such that each ray in Δ is ϵ-dominated in G by its origin, and no odd cut of G separates two tails of a double-ray belonging to Δ, then G is also cycle-decomposable.*

Proof Suppose the contrary. Then by Nash-Williams's Theorem, G has an odd cut $[A, \overline{A}]_G$. This implies that there exists a fragment $C \in \Delta$ which contains an odd number of edges of $[A, \overline{A}]_G$. However, it is easy to see that C cannot be a cycle, that if C is a double-ray, then $[A, \overline{A}]_G$ must separate two tails of C, and finally that if C is a ray, $[A, \overline{A}]_G$ must separate its origin from some of its tails. Hence in each of the three cases we contradict the hypothesis. $\square$

Definition 2.29 Given two sets Θ and Θ', both composed of edge-disjoint circuits and rays, we say that Θ' *has tails in* Θ if each tail of each infinite fragment of Θ' has a tail contained in some fragment of Θ.

Lemma 2.30 *Let $u \in V(G)$ and Δ be a decomposition of G into circuits and u-rays. Then every finite set Θ' of edge-disjoint circuits and u-rays of G that has tails in Δ can be extended to a decomposition Δ' of G into circuits and u-rays such that Δ has tails in Δ' and Δ' has tails in Δ.*

Proof Without loss of generality we may suppose that the Δ-shadow of $\bigcup_{C \in \Theta'} C$ is G itself. This implies, since Θ' is finite and has tails in Δ, that Δ is finite, and therefore that G is locally finite. Let Θ'' be a maximal set of edge-disjoint u-rays and double-rays of $G \setminus \bigcup_{C \in \Theta'} C$ that has tails in Δ. Put $H := \bigcup_{C \in \Theta' \cup \Theta''} C$. It is easy to see that no tail of any infinite fragment of Δ is contained in $G \setminus H$. Hence, Δ being finite and G locally finite, $G \setminus H$ is therefore an eulerian graph having no infinite component. Thus $G \setminus H$ has a cycle decomposition and any such cycle decomposition together with Θ' and Θ'' form a decomposition Δ' of G that has the desired properties. $\square$

Lemma 2.31 *Let G be a connected graph, $x \in V(G)$, and Δ be a decomposition of G into cycles and exactly two x-rays. Then, for every $v \in V(G)$, there exists a decomposition of G into cycles and exactly two v-rays.*

Proof Let R and R' be the two x-rays of Δ, P be any xv-path of G, and C_1, C_2, $\ldots$ C_n the cycles of Δ that edge-intersect P. Put $H := R \cup R' \cup C_1 \cup C_2 \cup \ldots \cup C_n$. Since H is an infinite locally finite connected eulerian graph, it contains a v-ray Q. Since $H \backslash Q$ is locally finite and has exactly one vertex of odd degree (viz. v), it contains a v-ray Q'. Since $H \backslash (Q \cup Q')$ is eulerian and locally finite, by Theorem 2.24, it has a circuit decomposition Δ'.

Now, observe that H is the edge-disjoint union of R_1, R_2 and the finite graph $C_1 \cup C_2 \cup \ldots \cup C_n$. This implies that $\{Q, Q'\}$ has tails in $\{R, R'\}$, and therefore that $H \backslash (Q \cup Q')$ is finite. Thus, Δ' is a cycle decomposition, and

$$\Delta'' := \{Q, Q'\} \cup \Delta' \cup (\Delta \backslash \{C_1, C_2, \ldots, C_n, R, R'\})$$

is the desired decomposition of G. $\square$

3 Rays in cycle-decomposable graphs

In this section we look for rays that can be removed from (or added to) a cycle-decomposable graph without affecting its cycle-decomposability.

Proposition 3.1 *Let H be a cycle-decomposable graph, R a ray which is edge-disjoint from H and ϵ-dominated by its origin in $H \cup R$. Then $H \cup R$ is likewise cycle-decomposable.*

Proof As in the proof of Proposition 2.28, suppose the contrary and consider an odd cut $[A, \overline{A}]_{H \cup R}$. Since the cut $[A \cap H, \overline{A} \cap H]_H$ must be of even cardinality, $E(R)$ must meet $[A, \overline{A}]_{H \cup R}$ an odd number of times, implying that a tail of R is separated from its origin by an odd cut of $H \cup R$, contradicting the hypothesis. $\square$

Hence in a decomposition into cycles and rays, the rays that are ϵ-dominated by their origin can in some sense be “melted" into the cycle-decomposable part. Conversely, one may ask if the removal of such rays from a cycle-decomposable graph will leave the remainder with a cycle decomposition. The answer in general is no; nevertheless there exist rather wide sufficient conditions, in particular Theorem 3.3, that allow one to find such “removable" rays. Note that such a ray must always be ϵ-dominated by its origin because otherwise the cycle-decomposable graph would have an even cut that separates the origin of the ray from some of its tails. On the other hand, as is shown by the next proposition, we may be able to “remove” such rays in pairs.

Proposition 3.2 *Let G be a cycle-decomposable connected graph, $u \in V(G)$, and τ be a non-ϵ-dominated end of G. Then there exist two edge-disjoint u-rays S_1 and S_2 of τ such that $G \backslash (S_1 \cup S_2)$ is still cycle-decomposable.*

Proof Let $R \in \tau$ be a u-ray and H be the Δ-shadow of R, where Δ is some cycle decomposition of G. Then H must be locally finite and every ray of H is end-equivalent to R in H. Hence there is no finite set of edges of H whose deletion from H leaves more than one infinite component, which implies by Theorem 2.1 that H is a 2-way infinite trail, and hence is decomposable into two edge-disjoint 1-way infinite trails P_1 and P_2, both starting at u. Being locally finite, each of the trails P_i is decomposable into cycles and exactly one u-ray (say S_i). Hence $H\backslash(S_1 \cup S_2)$ is cycle-decomposable, and therefore so is $G\backslash(S_1 \cup S_2)$. $\square$

The next theorem shows that in most ϵ-dominated ends, there exist rays that can be removed, without affecting the cycle-decomposability of the graph.

Theorem 3.3 *Let G be a cycle-decomposable graph, $u \in V(G)$, $n \in \omega$ and τ be an end of ϵ-multiplicity $> n$ that is ϵ-dominated by u in G. Then G admits a decomposition into cycles and exactly n u-rays belonging to τ.*

The proof is based on the two following lemmas.

Lemma 3.4 *Let G be a cycle-decomposable graph, $u \in V(G)$, and $\{R, R'\}$ be a 2-tresse that consists of two u-rays R and R', both ϵ-dominated by u. Let $K := R \cup R'$, then there exists a u-ray $S \subseteq K$ such that $G\backslash S$ is still cycle-decomposable.*

Moreover, given any countable family $(R_i)_{i \in I}$ of edge-disjoint rays of $G\backslash K$, that are end-equivalent to R (and hence to R') in G, S can be so chosen that there exists a u-ray $S' \subseteq K\backslash S$ which is both ϵ-dominated by u and end-equivalent to the R_i's in $G\backslash S$.

Proof Let τ be the end of G that contains R, R' and the R_i's. Note that we may suppose that G is countable, because otherwise we could take any countable subgraph of G in which R, R' and the R_i's are still end-equivalent and ϵ-dominated by u, and let H be the shadow of this subgraph with respect to some cycle decomposition of G. It is easy to see that if the result holds for H, it will also hold for G, because H and $G\backslash H$ are both cycle-decomposable, and end-equivalence and ϵ-domination in H imply the same properties in G.

We may furthermore assume that any ray of τ edge-intersects K or some R_i, because otherwise we could add additional rays to the family R_i, $i \in I$, to obtain a maximal one. Since G is assumed to be countable, the extended family will also be countable.

Now let $U = \{u_j\}_{j \in J}$ be the set of all vertices that ϵ-dominate τ in G. Note that J is countable and suppose for convenience that the index sets J and I are disjoint.

For each $i \in I$, let $(Q_i^n)_{n \in \omega}$ be an infinite family of vertex-disjoint R_iK-paths, and for each $j \in J$, let $(Q_j^n)_{n \in \omega}$ be a family of edge-disjoint u_jK-paths whose endpoints in K are all distinct (see Figure 2).

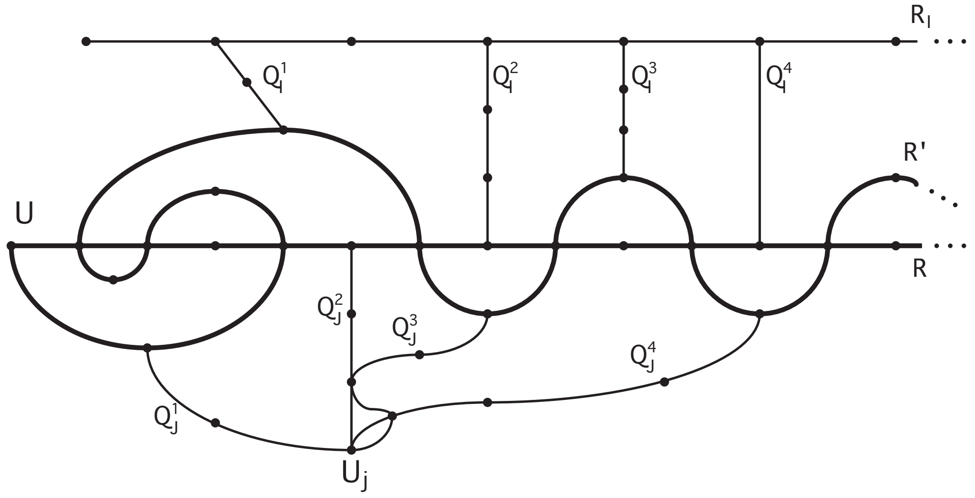

Figure 2:

Since both R and R' are end-equivalent to each R_i and ϵ-dominated by each u_j in G, such (Q_i^n)'s exist for any $i \in I \cup J$, and without loss of generality may be assumed to be edge-disjoint from K $(= R \cup R')$.

Let $X = \{x_m\}_{m\in\omega}$ be an infinite set of vertices of $V(R) \cap V(R')$ such that $x_0 = u$ and, for any $m \in \omega$, x_m is closer to u than x_{m+1} on both R and R'. Such a family is easily constructed by taking as x_1 any vertex of $V(R) \cap V(R') - \{u\}$, and as x_2, any vertex of $V(R) \cap V(R')$ which is beyond x_1 on both R and R', etc.

Choose any function $\phi : \omega \to I\dot{\cup}J$ for which each element of the range is the image of infinitely many elements of the domain. We will now recursively define two nested families $(W_i)_{i\in\omega}$ and $(W'_i)_{i\in\omega}$ of uX-trails contained in $R \cup R'$ having the following properties: there exists an increasing sequence $m_0 < m_1 < m_2 < \dots$ of non-negative integers such that, for each $i \in \omega$, $E(W_{i+1}) - E(W_i)$ and $E(W'_{i+1}) - E(W'_i)$ are (not necessarily respectively) the sets of edges of the $x_{m_i}x_{m_{i+1}}$-segments of R and R'.

Set $W_0 = W'_0 = \langle u\rangle$, where $\langle u\rangle$ is the path whose only vertex is u, and suppose W_k and W'_k have already been defined. Let x_{m_k} be the vertex $\neq u$ which is an endpoint of both W_k and W'_k. Let n_k be the smallest n for which the endpoint of $Q^n_{\phi(k)}$ on $R \cup R'$ does not belong to W_k or W'_k. Denote this endpoint by y_k and let m_{k+1} be the smallest subscript m for which y_k belongs to the $x_{m_k}x_m$-segment of R or of R', as the case may be. Denote by P_k the $x_{m_k}x_{m_{k+1}}$-segment of R and by P'_k the corresponding segment of R'.

If y_k belongs to P_k, then define $W_{k+1} := W_k \cup P'_k$ and $W'_{k+1} := W'_k \cup P_k$, and otherwise define $W_{k+1} := W_k \cup P_k$ and $W'_{k+1} := W'_k \cup P'_k$. Finally put $W := \bigcup_{k\in\omega} W_k$ and $W' := \bigcup_{k\in\omega} W'_k$.

It is easy to see that both W and W' are edge-disjoint 1-way infinite trails starting at u and that $W \cup W' = R \cup R'$. Hence W and W' are locally finite. Denote by $\langle u = x_0,\, e_0,\, x_1,\, e_1,\, \dots\rangle$ the 1-way infinite trail W, and let $S := \langle u = x_{i_0}, e_{i_0}, x_{i_1}, e_{i_1}, \ldots \rangle$ be the u-ray defined such that, i_0 is the largest index i for which $x_i = u$ and such that i_{j+1} is the largest index i for which $x_i = x_{i_j+1}$ (such indices exists because W is locally finite). Hence, for each $j \in \omega$, the sub-trail $\langle x_{i_j+1}, e_{i_j+1}, x_{i_j+2}, e_{i_j+2}, \ldots, x_{i_{j+1}} \rangle$ induce an eulerian finite subgraph of W edge-disjoint from S. In fact, $W \backslash S$ is exactly the edge-disjoint union of all such induced subgraphs. Since, by Theorem 2.24, each of these finite subgraphs is circuit-decomposable, $W \backslash S$ is also cycle-decomposable. Similarily, one can define an u-ray $S' \subseteq W'$ such that $W' \backslash S'$ is cycle-decomposable. Now, let us prove that S and S' have the desired properties. To do so we will show that:

(1) S' and R_i are end-equivalent in $G \backslash W$ for any $i \in I$;

(2) u_j ϵ-dominates S' in $G \backslash W$ for any $u_j \in U$;

(3) $G \backslash W$ is cycle-decomposable.

Clearly, the result follows from Properties (1), (2) and (3) because $W \backslash S$ is cycle-decomposable, $G \backslash W \subseteq G \backslash S$, and $u \in U$.

By the construction of W', for any $i \in I$, there exist infinitely many Q_i^n's for which one endpoint belongs to W'. This together with the fact that W' is a locally finite 1-way infinite trail implies that S' is end-equivalent to R_i in $G \backslash W$. We leave the details to the reader. For similar reasons, one can also conclude that any u_j, $j \in J$, ϵ-dominates S' in $G \backslash W$.

For the third statement, suppose the contrary. By Nash-Williams's Theorem, there is an odd cut $[A, \overline{A}]_{G \backslash W}$ with $u \in V(A)$; and clearly we can suppose $\overline{A}$ to be connected. Since $u \in U$, we infer from (2) that A contains a tail of S', and therefore by (1) and (2), A contains U as well as a tail of each R_i. Moreover, it is easy to see that since G contains no odd cuts, $Z := V(W) \cap V(\overline{A})$ is infinite. Applying Proposition 2.11 with $X := Z$ and $G := \overline{A}$, we obtain either a vertex $x \in V(\overline{A})$ and an infinite family of xZ-paths of $\overline{A}$ which pairwise intersect in x only, or a ray $S_0 \subseteq \overline{A}$ and infinitely many disjoint S_0Z-paths. In the first case, since W is a locally finite 1-way infinite trail that contains S, and since x cannot be separated from any infinite 1-way infinite sub-trail of W by the removal of a finite number of vertices of $W \cup \overline{A}$, we have that x cannot be separated from a tail of S by such a removal. Thus x must dominate S in $W \cup \overline{A}$ and hence in G, contradicting the fact that A contains U, which contains all vertices that dominate S. In the second case, again because W is a locally finite 1-way infinite trail that contains S, similarly as for the first case, we have that the removal of a finite number of vertices of $W \cup \overline{A}$ cannot separate a tail of S_0 from a tail of S. Thus, S_0 is end-equivalent to S in $\overline{A} \cup S$ (and therefore in G), whence $S_0 \in \tau$. Since A contains a tail of S' and of each R_i and since $[A, \overline{A}]_{G \backslash W}$ is finite, it follows that $V(\overline{A}) \cap V(S' \cup \bigcup_{i \in I} R_i)$ is finite. Moreover, since S' is contained in the one-way infinite trail W', $V(\overline{A}) \cap V(W' \cup \bigcup_{i \in I} R_i)$ is also finite. Since $S_0 \in \tau$ is contained in $\overline{A}$, some tail of it is edge-disjoint from all R_i's, from W' and since $S_0 \subseteq \overline{A} = (G \backslash W) - A$, also edge-disjoint from W. Again this is a contradiction because it has been assumed that every ray of τ edge-intersects K $(= R \cup R' = W \cup W')$

or some R_i. $\square$

Lemma 3.5 *Let τ be any end of G. If a cycle-decomposable locally finite subgraph H of G contains at least one ray of τ, then H contains a 2-tresse composed of rays of τ.*

Proof Let R_0 be any ray of τ contained in H, and J be the shadow of R_0 with respect to some cycle decomposition of H. This implies that no finite subset of $E(J)$ separates any ray of J from R_0 and so, since J is locally finite, no finite subset of $V(J)$ separates any ray of J from R_0. Hence all rays of J are end-equivalent in J. By Corollary 2.9, J contains a ray R that meets every other ray of J infinitely often. Since $J \backslash R$ is locally finite and has exactly one vertex of odd degree, it must contain a ray which together with R forms the desired pair of rays. $\square$

Proof of Theorem 3.3

Claim 1: *Without loss of generality we may suppose that G is countable, and that every vertex of infinite degree ϵ-dominates τ.* Let H_1 be the union of some $n+1$ edge-disjoint rays $S_1, \ldots, S_{n+1}$ of τ, extend H_1 to a new subgraph H_2 by adding sufficiently many pairwise vertex-disjoint H_1-paths so that the S_i's all are end-equivalent in H_2. It is easy to see that such paths exist because the S_i's are end-equivalent in G. Note that H_2 is connected, is locally finite and has exactly one end. Then extend H_2 to H_3 by adding countably many uH_2-paths which are pairwise edge-disjoint and edge-disjoint from H_2 such that the S_i's are ϵ-dominated by u in H_3. Finally, let H_4 be the shadow of H_3 with respect to some cycle decomposition of G. Clearly H_4 is cycle-decomposable. Moreover, by construction, no vertex of infinite degree in H_4 may be separated from u by a finite cutset of H_4. Since u ϵ-dominates the S_i's in H_4, this implies that any vertex of infinite degree of H_4 also ϵ-dominates the S_i's in H_4. Then it is easy to see that H_4 is countable, satisfies the conditions of the theorem (for the end τ_4 that contains the S_i's) and that if the conclusion of the theorem holds for H_4 with respect to τ_4, it will hold for G with respect to τ. So without loss of generality we may suppose that $G = H_4$.

From now on, suppose by way of contradiction, that G admits no decomposition into cycles and exactly n u-rays belonging to τ. Suppose that n is the smallest integer for which there exists a graph that satisfies the conditions of the theorem but not its conclusions. Since such a graph is cycle-decomposable, $n > 0$.

Claim 2: *Given any $n+1$ pairwise edge-disjoint rays of τ, no two of these rays form a 2-tresse.* By way of contradiction, suppose there exist edge-disjoint rays $R_1, R_2, \ldots, R_{n+1}$ of τ such that R_1 and R_2 meet each other infinitely often. Since the R_i's are ϵ-dominated by u in G, we may suppose without loss of generality that they originate at u. Put

$$R := R_1, R' := R_2 \text{ and } I := \{3, 4, \ldots, n+1\},$$

and let S and S' be the two u-rays that are given in Lemma 3.4. Thus the end τ' of $G\backslash S$ which contains the ray S' is ϵ-dominated by u in $G\backslash S$ and has a ϵ-multiplicity $> n-1$. Also by Lemma 3.4, $G\backslash S$ is cycle-decomposable; therefore, by the minimality of n, $G\backslash S$ has a decomposition into cycles and exactly $n-1$ u-rays belonging to τ', implying that G has the required decomposition.

Denote by D the set of vertices that dominate τ in G. Then D is finite by Claim 2 and Remark 2.4.

Claim 3: *Let G' be any (multi-)graph satisfying the conditions of the theorem and the conclusions of Claim 1 and 2. If $\{R_i\}_{i\in I}$ is a finite set of at least n edge-disjoint u-rays of τ, then every odd region of $G\backslash(\bigcup_{i\in I} R_i)$ meets D.*

By way of contradiction, suppose that A is an odd region of

$$G_1 := G\backslash(\textstyle\bigcup_{i\in I} R_i),$$

which is disjoint from D. Then there are two cases to consider.

Case 1. There exists a vertex $v \in V(A)$ which is of infinite degree in A. Since v is also of infinite degree in G, v ϵ-dominates τ in G (Claim 1). This implies that there exists $j \in I$ such that v ϵ-dominates R_j in $R_j \cup A$. However, since $V(A) \cap D = \emptyset$, R_j is not dominated in $R_j \cup A$. Thus, by Proposition 2.19 (i)$\Rightarrow$(iii), it is easy to see that $R_j \cup A$ contains at least two edge-disjoint rays S_1, S_2 that meet each other infinitely often, and are end-equivalent to R_j in $R_j \cup A$ (and hence in G). Hence, S_1, S_2 together with the R_i's for $i \neq j$ contradict Claim 2.

Case 2. A is locally finite. Let A^+ be the subgraph obtained from A by adding the edges in $[A, \overline{A}]_{G_1}$, and let $\theta = (\theta_x)_{x\in V(G)}$ be the transition system obtained from some cycle decomposition of G. Note that A^+ also is locally finite. For each $x \in V(A)$, let θ'_x be the set of all pairs $\{e, e'\} \in \theta_x$ such that $e, e' \in E(A^+)$. Since u is of infinite degree in G (and hence in G_1), $u \notin V(A)$. Thus, given any $x \in V(A)$, the degree of x in $\bigcup_{i\in I} R_i$ is even (possibly zero). Moreover, the edges of G_1 incident with x which do not belong to any pair of θ'_x are precisely those which are coupled in θ_x with an edge of some R_i. Since θ_x is a pairing of all the edges of G incident with x, it follows that the number of such edges which belong to G_1 (i.e. to A^+) but do not belong to any pair of θ'_x is even. Thus, θ'_x can be extended to a (full) pairing θ''_x of the set of edges of A^+ incident with x. Hence, the pairings θ''_x, $x \in V(A)$, induce a decomposition Θ'' of A^+ into edge-traceable graphs, finite non-eulerian trails and 1-way infinite trails. Note that the last two categories of fragments of Θ'' must have their initial and (in the case of finite non-eulerian trails) terminal edges in $[A, \overline{A}]_{G_1}$. Since $[A, \overline{A}]_{G_1}$ is of odd cardinality, it follows that Θ'' must contain a 1-way infinite trail K. The transition system θ being induced by a cycle decomposition, it is easy to see that K must meet $\bigcup_{i\in I} R_i$ (and hence some R_{i_0}) infinitely often.

Let x_0 be any vertex of $V(K) \cap V(R_{i_0})$ and H the union of the tail of R_{i_0} that begins at x_0 and a 1-way infinite subtrail of K that also starts at x_0. Being a subgraph of $A^+ \cup R_{i_0}$, H is locally finite. By construction, it is one-ended, and hence cannot have any odd cuts. This implies by Nash-Williams's Theorem that H is cycle-decomposable. By Lemma 3.5, H must contain two rays S_1 and S_2 which meet infinitely often and are end-equivalent to R_{i_0} in G. Thus S_1, S_2 and $\{R_i\}_{i\in I-\{i_0\}}$ again contradict Claim 2. This proves Claim 3.

By Nash-Williams's Theorem it is easy to see that a graph which is not cycle-decomposable must contain two vertex-disjoint odd regions. Hence, if $R_1, \dots, R_n$ are edge-disjoint u-rays of τ, then by Claim 3 there exist two vertices in D which are not infinitely edge-connected in $G \backslash \bigcup_{i=1}^{n} R_i$. Thus to finish the proof, it suffices to construct n pairwise edge-disjoint u-rays $S_1, \dots, S_n$ of τ such that the vertices in D are still pairwise infinitely edge-connected in $G \backslash (\bigcup_{i=1}^{n} S_i)$.

Let $R_1, \dots, R_{n+1}$ be pairwise edge-disjoint u-rays of τ and

$$L := \bigcup_{i=1}^{n+1} R_i.$$

We will now show that there exists a decomposition Δ of $G \backslash L$ into cycles and a finite number of D-paths. It clearly suffices to find a decomposition of $G \backslash L$ into cycles and an arbitrary number (possibly infinite) of D-paths. Denote by G' the multigraph obtained by adding to G a new vertex w and, for each $x \in D$, a countably infinite set of xw-edges. Since D is finite, the vertices that dominate the R_i's in G' are exactly the vertices of D (w does not dominate them). Also, in $G' \backslash L$ all the vertices of D are still infinitely edge-connected. Moreover, $G' \backslash L$ does not have any odd cut because otherwise G will contradict Claim 3. Hence by Nash-Williams's Theorem, $G' \backslash L$ has a cycle decomposition and by removing the edges incident with w, we obtain the desired decomposition of $G \backslash L$.

Now, choose a family $(Q_k)_{k \in \omega}$ of vertex-disjoint L-paths, and for each $v \in D$, a family $(P^v_m)_{m \in \omega}$ of vL-paths pairwise intersecting in v only such that

(a) the R_i's are end-equivalent in $L \cup (\bigcup_{k \in \omega} Q_k)$;

(b) each P^v_m is edge-disjoint from $L \cup (\bigcup_{k \in \omega} Q_k)$;

(c) each Q_k is internally vertex-disjoint from L;

(d) the Δ-shadows of the Q_k's are pairwise edge-disjoint and edge-disjoint from all the D-paths of Δ.

We leave it to the reader to show that since both D and the fragments of Δ are finite, one can recursively construct such families.

Denote by E_v the set of all edges of L that have a vertex in common with some P^v_m, $m \in \omega$. Note that E_v is infinite for any $v \in D$.

Finally let R be any u-ray in $L \cup (\bigcup_{k \in \omega} Q_k)$ such that:

(1) – R contains infinitely many edges of E_v for every $v \in D$,

(2) – if we orient the edges of each R_i in the natural way (i.e. away from u), then this will be consistent with the natural orientation of the edges of R.

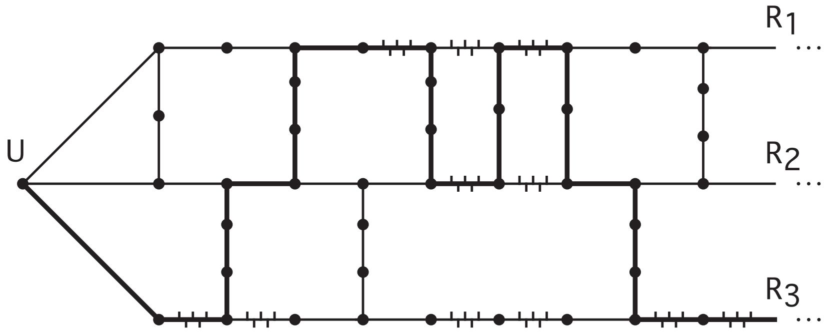

Figure 3: Edges of R: bold; edges belonging to some E_v: whiskered

(See Figure 3 for a simple example).

We leave it to the reader to show that such a ray exists and that it follows from Property (2) that the symmetric difference of R and L contains n edge-disjoint u-rays $S_1, \ldots, S_n$.

To finish the proof, let us show that any two vertices a, b of D are infinitely edge-connected in $G \setminus \bigcup_{i=1}^n S_i$. Observe that since both $E_a \cap E(R)$ and $E_b \cap E(R)$ are infinite, and since L is locally finite, a and b must be infinitely edge-connected in $(G \setminus L) \cup R$. Define R' to be the 1-way infinite trail obtained from R by replacing each $Q_k \subseteq R$ by the corresponding trail that is induced by the edges of the Δ-shadow of Q_k which do not belong to Q_k. Such an R' exists because the Q_k's are assumed to be pairwise vertex-disjoint and their Δ-shadows pairwise edge-disjoint and edge-disjoint from the D-paths of Δ.

Since both R and R' coincide on L and since E_a and E_b are contained in $E(L)$, a and b are also infinitely edge-connected in

$$G_1 := \Big((G \setminus L) \setminus \bigcup_{k \in \omega} Q_k\Big) \cup R'.$$

Moreover R' is edge-disjoint from $\bigcup_{k \in \omega} Q_k$ since by the choice of R, each Q_k is either contained in R or edge-disjoint from it. Thus $G_1 \subseteq G \setminus \bigcup_{i=1}^n S_i$ and we are done because a and b are then infinitely edge-connected in $G \setminus \bigcup_{i=1}^n S_i$. $\square$

Observe that, given a cycle-decomposable graph G and an end of ϵ-multiplicity $n+1$ that is ϵ-dominated by a vertex u, the maximal integer m for which there exists a decomposition of G into cycles and exactly m u-rays belonging to τ is at most $n+1$; from this point of view Theorem 3.3 says that $m = n$ or $n+1$. Our next proposition shows that $m = n+1$ when τ has exactly one dominating vertex.

Proposition 3.6 *Let G be a cycle-decomposable graph, $n \in \omega$, and τ be an end of ϵ-multiplicity $\geq n$ that has a unique dominating vertex u. Then for any vertex v that ϵ-dominates τ, G also admits a decomposition into cycles and exactly n v-rays belonging to τ.*

Proof By Theorem 3.3, we may assume that τ has ϵ-multiplicity exactly n. If $u \neq v$, note that since u and v are infinitely edge connected and ϵ-dominate τ, it is easy to see that we only have to show the result for the case where $v = u$. Let $R_1, \ldots, R_n$ be n pairwise edge-disjoint u-rays of τ, and H the Δ-shadow of $R_1 \cup \ldots \cup R_n$, where Δ is some cycle decomposition of G.

We claim that $L := H \backslash (R_1 \cup \ldots \cup R_n)$ is cycle-decomposable. Note that this claim implies the result because any cycle decomposition of L together with the R_i's and the cycles of Δ that are edge-disjoint from L form a decomposition of G that has the desired property. Thus, to finish the proof, suppose by way of contradiction that L is not cycle-decomposable. By Nash-Williams' theorem, there exists an odd region A of L that does not contain u. Since H is cycle-decomposable, $[A, H - A]_H$ cannot be of odd cardinality and hence must be infinite. This implies that $R_1 \cup \ldots \cup R_n$ (and hence some R_i) meets A infinitely often. Then by Proposition 2.12, A must contain a ray Q that is end equivalent to R_i in G. Hence $Q, R_1, \ldots, R_n$ are $n + 1$ pairwise edge-disjoint rays of τ, implying that τ has ϵ-multiplicity $> n$, a contradiction.

$\square$

From another point of view, Theorem 3.3 also says that for each $k \leq m$, G admits a decomposition into cycles and exactly k u-rays belonging to τ. The next proposition has a somewhat similar character.

Proposition 3.7 *Let $u \in V(G)$ and Δ be a decomposition of G into circuits and u-rays such that every ray contained in an infinite fragment of Δ is ϵ-dominated by u in G. Let r, s be respectively the number of rays and double-rays in Δ. Then, for any non-negative integer $n \leq r + 2s$, there exists a decomposition of G into cycles and exactly n u-rays, each ray of which is ϵ-dominated by u in G.*

Proof By way of contradiction suppose that n_0 is the smallest value of n for which there exists a counterexample and that G is such a counterexample.

By Proposition 2.28, G is cycle-decomposable, and therefore $n_0 > 0$.

Split each double-ray of Δ into two rays having the same origin, and let Ψ be the set of all these rays together with all the rays of Δ. Recall that $|\Psi| \geq n_0$. By Theorem 3.3, no end which is ϵ-dominated by u in G contains more than n_0 edge-disjoint rays. Hence u does not ϵ-dominate any ω-tresse, and it therefore follows from Proposition 2.22 that the partial order $(\Psi, \preceq_u)$ has a maximal element, say R. Thus, every ray of $\Psi - \{R\}$ is still ϵ-dominated by u in $G \backslash R$. Let R' be any u-ray that shares a tail with R, put $\Theta' := \{R'\}$, and let Δ' be a decomposition given by Lemma 2.30. Since the number of rays plus twice the number of double-rays is the same in Δ and Δ', it follows that $\Delta' - \{R'\}$ is a decomposition of $G \backslash R'$ into circuits and u-rays that satisfies the condition of the proposition for $n = n_0 - 1$. Hence $G \backslash R'$ has a decomposition Δ' into cycles and exactly $n_0 - 1$ u-rays ϵ-dominated by u in $G \backslash R'$. Since R' is ϵ-dominated by u in G, the decomposition $\Delta'' := \Delta' \cup \{R'\}$ of G gives rise to a contradiction. $\square$

Proposition 3.7 together with Theorem 3.3 and the next result, although they deal with cycle-decomposable graphs only, will be key results in our study of the circuit-decomposability of arbitrary graphs.

Lemma 3.8 *Let G be a cycle-decomposable connected graph and $v \in V(G)$. If G has a circuit decomposition that contains at least one double-ray, then G has a decomposition into cycles and exactly two v-rays.*

Proof First note that, G being connected, by Lemma 2.31 it is enough to show that there exists some vertex $x \in V(G)$ for which G admits a decomposition into cycles and exactly two x-rays. Let Δ_0 be any cycle decomposition and Δ_1 any circuit decomposition having at least one double-ray. There are two cases to consider.

Case 1. There is a region B of G that contains a tail of some fragment of Δ_1, and there exists a vertex $x \in V(G)$ that ϵ-dominates all tails contained in B of infinite fragments of Δ_1. (Note that $x \in V(B)$.) By Nash-Williams's Theorem $[B, \overline{B}]_G$ is an even cut. Thus the quotient graph $G/\overline{B}$ has a decomposition into circuits (essentially induced by Δ_1) which satisfies the conditions of Proposition 3.7 for $n = 2$, i.e., $G/\overline{B}$ has a decomposition Θ into cycles and exactly two x-rays, each x-ray of Θ being ϵ-dominated by x in $G/\overline{B}$. Since the number of edges of $G/\overline{B}$ incident with $q_{\overline{B}}$ is $|[B, \overline{B}]_G|$ (which is finite), and since x ϵ-dominates the two rays of Θ, B contains two edge-disjoint x-rays that have tails in Θ. Therefore by Lemma 2.30, Θ may be so chosen that its two x-rays do not contain $q_{\overline{B}}$. Define Δ as the set of all fragments of Θ that do not contain $q_{\overline{B}}$ together with all fragments of Δ_0 that are contained in $\overline{B}$. Observe that every fragment of Δ (even those that belong to Θ) may be considered as a subgraph of G, and that $G \backslash (\bigcup_{C \in \Delta} C)$ is cycle-decomposable because it is eulerian and contains only a finite number of edges. Thus Δ can be extended to a decomposition of all of G into cycles and exactly two x-rays.

Case 2. G contains no region having the properties of Case 1. Denote by $\mathcal{B}$ the set of all regions of G that contain a tail of some infinite fragment of Δ_1. For each $B_0 \in \mathcal{B}$ and $y \in \mathrm{Bdry}_G(B_0)$ some fragment of Δ_1 contains a ray $R_0 \subseteq B_0$ which is not ϵ-dominated by y. Hence there exists a region $B' \in \mathcal{B}$ such that $B' \subseteq B_0 - y$.

Since $\mathrm{Bdry}_G(B_0)$ is finite, this implies that there exists a region $B_1 \in \mathcal{B}$ which is contained in B_0 and is disjoint from $\mathrm{Bdry}_G(B_0)$. Repeating this argument, we obtain an ϵ-stratifying sequence $(B_n)_{n \in \omega}$ of G. Thus, by Proposition 2.17, G has a non-ϵ-dominated end, and therefore by Proposition 3.2, the required decomposition of G exists. □

4 Having enough rays, a necessary condition

By Nash-Williams's Theorem, a graph without odd cuts has a cycle decomposition and hence a circuit decomposition. Moreover, for any odd cut $[A, \overline{A}]_G$ of a circuit-decomposable graph G, there must exist a transverse double-ray, i.e. a double-ray having a tail in A and another one in $\overline{A}$. This is because given any circuit decomposition Δ of G, the cyles belonging to Δ must meet $[A, \overline{A}]_G$ an even number of times, and the same holds for the non-transverse double-rays in Δ. This observation leads us to the following definition.

Definition 4.1 We say that a graph G has *enough rays* if for each odd cut $[A, \overline{A}]_G$ of G, both A and $\overline{A}$ contain a ray.

It will be convenient to relativize this definition to various specific classes of rays (e.g. non-dominated rays, removable rays) in the obvious way.

In view of the symmetry of this definition and since for any odd cut $[A, \overline{A}]_G$, A has only finitely many components that meet $\mathrm{Bdry}_G(A)$, each being a region of G, not all of them even, the definition can be equivalently stated as: a graph G has *enough rays* if each odd region contains a ray.

In this language the remark at the beginning of this section becomes the following necessary condition:

Proposition 4.2 *A circuit-decomposable graph has enough rays.*

Observe that this necessary condition implies that the graph is eulerian, because for any vertex of odd degree x of G, $[x, G-x]_G$ is an odd cut and clearly x does not contain rays. Unfortunately the condition stated in Proposition 4.2 is not sufficient: see Figure 4.

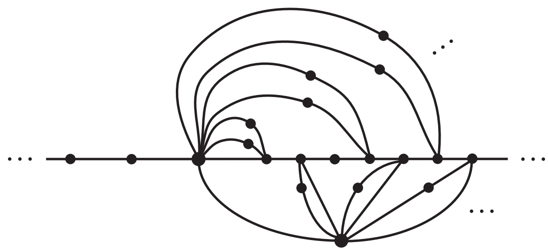

Figure 4:

The example of Figure 4 contains two vertices of infinite degree; this is the minimal number where the condition is not sufficient since, as will show Proposition 9.12, for graphs with at most one vertex of infinite degree the condition is both necessary and sufficient. This, incidentally, shows that the condition of having enough rays is strictly stronger than being eulerian.

5 Dominated subgraphs

Before going further, we need to generalize the domination property which so far is defined for rays only, to arbitrary subgraphs.

Definition 5.1 Let H be a subgraph of G and $x \in V(G)$. H is said to be *dominated* by x in G if there exists an infinite set of xH-paths pairwise intersecting in x only.

By Menger's Theorem, this definition has the following equivalent form:

Lemma 5.2 *A vertex x dominates a subgraph H in G if and only if for every finite set of vertices S of $V(G)\backslash\{x\}$ infinitely many vertices of H lie in the component of $G - S$ which contains x.*

Proof Left to the reader. $\square$

As we will show later, from the point of view of circuit-decomposability, one of the most interesting classes of subgraphs are the non-dominated eulerian subgraphs of G, because the removal of such a subgraph from G does not affect the decomposability or indecomposability of G into circuits (Proposition 8.1).

The next proposition provides a useful tool for proving whether or not a subgraph is dominated.

Lemma 5.3 *Let H be subgraph of a graph G. Then the following statements are equivalent:*

(i) *H is dominated in G;*

(ii) *G contains a rayless tree T such that $V(T \cap H)$ is infinite;*

(iii) *G contains a tree T and a vertex $x \in V(T)$ such that infinitely many different components of $T - x$ meet H.*

Proof (i)⇒(ii). Immediate from the fact that the union of an infinite family of xH-paths, pairwise intersecting in x only, is a rayless tree.

(ii)⇒(iii). Let F be any rayless tree meeting H infinitely often. For each pair $y, z \in V(H) \cap V(F)$, let P_{yz} be the unique yz-path in F, and let $T \subseteq F$ be the union of all P_{yz}'s. Note that T is still an infinite rayless tree because it is clearly connected, included in F and contains all vertices of $V(H) \cap V(F)$. Therefore some vertex x has infinite degree in T. Since T is a union of paths having their endpoints in H, every component of $T - x$ must contain a vertex of H, and since x is of infinite degree in T, there must be infinitely many such components.

(iii)⇒(i). It is easy to see that by taking one xH-path in each of the components of $T - x$ that meet H one obtains an infinite family of xH-paths pairwise intersecting in x only; i.e. H is dominated by x in G. $\square$

Observe that every subgraph H of a graph G containing a vertex x of infinite degree in H is dominated by x in G. On the other hand, a connected locally finite subgraph is or is not dominated depending on whether or not it contains dominated rays.

Lemma 5.4 *Let H be a subgraph of G that has a finite number of components. Then H is non-dominated in G if and only if H is locally finite and does not contain any dominated ray of G.*

Proof Without loss of generality we may suppose that both H and G are connected because otherwise we can prove the result for each component of H viewed as a subgraph of the component of G that contains it. Since H has only finitely many components, this will implies the result for the whole subgraph H.

The necessity is a straightforward consequence of the definition. For the sufficiency, suppose that H is dominated in G and locally finite; let us show that H must contain a dominated ray. By Lemma 5.3, let T be a rayless tree of G meeting H infinitely often, and let J be a spanning tree of H. Fix $x_0 \in V(H)$ and for each $x \in V(T) \cap V(H)$ let P_x be the x_0x-path in J. Then the union of these paths is infinite, connected and locally finite, and therefore must contain a ray R. R must meet infinitely many P_x's, and hence there exist infinitely many disjoint TR-paths in J. Clearly these paths can be chosen so that each includes just one vertex of T. Then the union of these paths and T is a rayless tree meeting R infinitely often, which by Lemma 5.3 gives the result. □

The following result relates the domination property of a subgraph to its cardinality.

Corollary 5.5 *Let G be a graph. Then*

(i) *Every finite subgraph of G is non-dominated in G.*

(ii) *Every non-dominated (not necessarily connected) subgraph of a connected graph G is countable and locally finite.*

Proof (i) is immediate from the definition of dominance. For (ii), let H be any non-dominated subgraph of G and T any spanning tree of G. Let $T' \subseteq T$ be the union of all paths of T having both endpoints in H. Clearly T' is a tree and since for any $x \in V(T')$, each component of $T' - x$ intersects H, Lemma 5.3 implies that T', and hence $T' \cup H$, is locally finite. Since $T' \cup H$ is connected, it follows that it is also countable. Thus H is likewise locally finite and countable. □

6 Having enough non-dominated rays, a sufficient condition

The aim of this section is to show that "having enough non-dominated rays" is a sufficient condition for a graph to have a circuit decomposition or more precisely that this property characterizes the graphs that have a decomposition into non-dominated circuits (Theorem 6.9). To do so we first have to restrict ourselves to the countable case; afterwards, we generalize to arbitrary cardinality using Theorem 6.8, which was proved in [11].

The following definition provides a generalization of the notion of "having enough rays".

Definition 6.1 A set $\mathcal{R}$ of ends of G is said to be *well-spread* in G if each odd region of G contains a ray belonging to $\mathcal{R}$.

By extension we say that $\mathcal{R}$ is *well-spread* in a subgraph H of G if each odd region of H contains a ray belonging to $\mathcal{R}$.

As already mentioned after Definition 4.1, a graph G has enough rays (resp. enough non-dominated rays) if and only if the set of all ends (resp. non-dominated ends) of G is well-spread.

The next lemma shows that the property of having enough non-dominated rays is resistant to the removal of non-dominated eulerian subgraphs.

Lemma 6.2 *Let $\mathcal{R}$ be a well-spread set of non-dominated ends of G, and H a locally finite eulerian subgraph of G that has at most finitely many connected components. If all rays contained in H belong to $\mathcal{R}$, then $\mathcal{R}$ is also well-spread in $G\backslash H$.*

Proof First note that by Lemma 5.4, H is not dominated in G. By way of contradiction, let us suppose that there is an odd region A of $G\backslash H$ that contains no ray of $\mathcal{R}$. There are two cases to consider.

Case 1. $V(H) \cap V(A)$ is infinite. Fix $x_0 \in V(A)$ and a spanning tree T of A and define T' as the subtree which is the union of all x_0H-paths of T. Since T' meets $V(H)$ infinitely often, Lemma 5.3 implies that T' contains a ray R.

Now let $\widehat{G}$ be a new graph obtained from G by the addition of a new vertex u joined to each vertex of R. It is easy to see that H is dominated by u in $\widehat{G}$. Since the number of components of H is finite, one of them, say K, is also dominated by u in $\widehat{G}$. Thus by Lemma 5.4, K must contain a ray Q which is dominated in $\widehat{G}$. Since by hypothesis Q is non-dominated in G, its dominating vertex in $\widehat{G}$ must be u, implying, by the construction of $\widehat{G}$, that Q and R belong to a same end of G. This contradicts the fact that Q, being a ray of H, belongs to $\mathcal{R}$ whereas R does not.

Case 2. $V(H) \cap V(A)$ is finite. By our notational convention, A is an induced subgraph of $G\backslash H$. Denote by A^+ the vertex-induced subgraph of G on $V(A)$, and put $\overline{A^+} := G - A^+$. Observe that $H \cap A^+$ must be finite because $V(A) = V(A^+)$ and $V(H) \cap V(A)$ is finite. Hence, H being locally finite, $[A^+, \overline{A^+}]_G$ is likewise finite. Moreover, $|E(H) \cap [A^+, \overline{A^+}]_G|$ is even because otherwise $[H \cap A^+, H \cap \overline{A^+}]_H$ would be an odd cut of H, and therefore, $G/\overline{A^+}$ a finite graph having exactly one vertex of odd degree, which is not possible.

To finish the proof, note that since

$$\Big|[A^+, \overline{A^+}]_G\Big| = \Big|[A, \overline{A}]_{G\backslash H}\Big| + \Big|E(H) \cap [A^+, \overline{A^+}]_G\Big|,$$

the cut $[A^+, \overline{A^+}]_G$ of G is odd, and since A $(= A^+\backslash H)$ contains no rays of $\mathcal{R}$ and $H \cap A^+$ is finite, A^+ is likewise free of rays of $\mathcal{R}$. This contradicts the hypothesis that $\mathcal{R}$ is well-spread in G. $\Box$

Lemma 6.3 *Let $\mathcal{R}$ be a well-spread set of non-dominated ends of G. Then every edge of G is contained in a circuit of G which is either finite or the union of two rays of $\mathcal{R}$.*

Proof We may suppose G to be connected since $\mathcal{R}$ is well-spread in each component of G. Let $e \in E(G)$. If e is contained in a cycle, then there is nothing to show. If this is not the case, then $\{e\}$ must be an odd cut, call it $[A, \overline{A}]_G$. Since $\mathcal{R}$ is well-spread, let R_A (resp. $R_{\overline{A}}$) be a ray of $\mathcal{R}$ that is contained in A (resp. $\overline{A}$). Since any two rays having a common tail are equivalent, we may suppose without loss of generality that the origin of R_A (resp. $R_{\overline{A}}$) is the only vertex of A (resp. $\overline{A}$) incident to e. Then $D := R_A \cup e \cup R_{\overline{A}}$ is a double-ray of G all of whose tails belong to $\mathcal{R}$, and e belongs to D. $\square$

Proposition 6.4 *Let G be a countable graph and $\mathcal{R}$ a well-spread set of non-dominated ends of G. Then G has a circuit decomposition Δ such that $\mathcal{R}$ is still well-spread in each fragment of Δ (i.e. each double-ray of Δ is the union of two rays of $\mathcal{R}$).*

Note that by Lemma 5.4, the circuits of such a decomposition are all non-dominated in G.

Proof Let $\{e_1, e_2, \ldots\}$ be an enumeration of $E(G)$. We will construct a circuit decomposition $(C_i)_{i \in I}$ inductively in the following way: suppose that for every positive integer $k < j$, C_k has already been constructed and that C_k is either finite or the union of two rays of $\mathcal{R}$. Let i_j be the smallest subscript such that $e_{i_j} \notin \bigcup_{k<j} E(C_k)$. If no such subscript exists then $(C_i)_{i<j}$ is the desired circuit decomposition. Otherwise, by Lemma 6.2, $\mathcal{R}$ is well-spread in $G \backslash (\bigcup_{k<j} C_k)$. By Lemma 6.3, let $C_j \subseteq G \backslash (\bigcup_{k<j} C_k)$ be any circuit which is either finite or the union of two rays of $\mathcal{R}$ and contains the edge e_{i_j}. Clearly $(C_k)_{k<r}$ is, for some $r \leq \omega$, a circuit decomposition of G that has the desired property. $\square$

The generalization of this last result to the uncountable case is an immediate consequence of the following key result.

Theorem 6.5 *Let $\mathcal{R}$ be a well-spread set of non-dominated ends of G. Then G is decomposable into countable fragments in each of which $\mathcal{R}$ is still well-spread.*

For the proof of this theorem we have to recall some definitions and a result from Part I of this paper [11].

Definition 6.6 An *α-decomposition* of a graph is a decomposition whose fragments are all of cardinality less than or equal to α.

Definition 6.7 An α-decomposition Δ of a graph G is said to be *bond-faithful* if

(i) any bond of G of cardinality $\leq \alpha$ is contained in some fragment of Δ;

(ii) any bond of cardinality $< \alpha$ of a fragment of Δ is also a bond of G.

Since any cut is the edge-disjoint union of bonds, condition (i) implies that condition (ii) can be replaced in an equivalent manner by:

(ii′) any cut of cardinality $< \alpha$ of a fragment of Δ is also a cut of G.

Theorem 6.8 **[11]** *Let $(H_i)_{i\in I}$ be a pairwise edge-disjoint family of countable connected subgraphs of G. Then G has a bond-faithful ω-decomposition Δ such that each H_i and each non-isolated vertex of degree $\leq \omega$ in G is contained in one and only one fragment of Δ.*

Assuming the Generalized Continuum Hypothesis, this last result can be generalized to α-decompositions with $\alpha > \omega$. See **[11]**.

Proof of Theorem 6.5 Let $\mathcal{U}$ be a maximal set of pairwise vertex-disjoint rays of $\mathcal{R}$. We claim that for every odd region A of G there exists a ray in $\mathcal{U}$ that has a tail in A. Suppose that an odd region A of G does not contain the tail of any ray in $\mathcal{U}$. Since $[A, \overline{A}]_G$ is finite, $V(R \cap A)$ is finite for every $R \in \mathcal{U}$. Moreover, since the rays in $\mathcal{U}$ are pairwise edge-disjoint, at most $|[A, \overline{A}]_G|$ rays in $\mathcal{U}$ meet A, and so $V(A \cap \bigcup_{R\in\mathcal{U}} R)$ is finite. Hence any ray in A has a tail which is disjoint from all rays in $\mathcal{U}$. This contradicts the maximality of $\mathcal{U}$, since A contains a ray belonging to $\mathcal{R}$, and any tail of a such a ray is likewise in $\mathcal{R}$.

Let $\widehat{G}$ be the graph obtained from G by the addition of a new vertex u joined to the origin of each ray in $\mathcal{U}$, and for any $R \in \mathcal{U}$, consider the ray

$$\widehat{R} := e_R \cup R,$$

e_R being the edge joining u to the origin of R. Let $\widehat{\Delta}$ be a bond-faithful ω-decomposition of $\widehat{G}$ such that for any $R \in \mathcal{U}$, $\widehat{R}$ is contained in a fragment of $\widehat{\Delta}$. Such a decomposition exists by Theorem 6.8, $(\widehat{R})_{R\in\mathcal{U}}$ being a family of pairwise edge-disjoint graphs of cardinality at most ω.

Claim: We may suppose that no fragment of $\widehat{\Delta}$ has u as a cut-vertex. Otherwise decompose each fragment $\widehat{H} \in \widehat{\Delta}$ into $(\widehat{H}_i)_{i\in I_H}$, where each $\widehat{H}_i$ is a component of $\widehat{H} - u$ together with u and all the edges of $\widehat{H}$ joining u to a vertex of that component. Note that if u is not a cut-vertex of $\widehat{H}$, then $|I_H| = 1$. Consider the ω-decomposition Ξ of $\widehat{G}$ whose fragments are all the $\widehat{H}_i$'s, where $\widehat{H}$ runs through all fragments of $\widehat{\Delta}$. Since every ray $\widehat{R}\,(R \in \mathcal{U})$ contains exactly one edge incident with u, it is easy to see that each such $\widehat{R}$ is still contained in one and only one fragment of Ξ.

In any graph, no two edges of a bond are separated by a cut-vertex. It follows that for each $\widehat{H} \in \widehat{\Delta}$ the set of all bonds of $\widehat{H}$ is exactly the set of all bonds of all the $\widehat{H}_i$'s ($i \in I_H$). Thus , Ξ is a bond-faithful decomposition of $\widehat{G}$, and therefore can play the role of $\widehat{\Delta}$, proving the Claim.

Now, let

$$\Delta := (\widehat{H} - u)_{\widehat{H}\in\widehat{\Delta}}.$$

where $\widehat{H} - u$ means $\widehat{H}$ if $u \notin V(\widehat{H})$. Clearly, Δ is an ω-decomposition of G since no fragment of $\widehat{\Delta}$ contains u as a cut-vertex. Let us prove that Δ is the desired decomposition of G (i.e. $\mathcal{R}$ is well-spread in each fragment of Δ).

Let H be any fragment of Δ, $\widehat{H}$ the corresponding fragment in $\widehat{\Delta}$ and $C = [A, H - A]_H$ any odd cut of H. To finish the proof, let us show that A contains a tail of a ray of $\mathcal{U}$. By way of contradiction suppose that no ray of $\mathcal{U}$ has a tail in A. This implies that at most a finite number of the rays of $\mathcal{U}$ which are contained in H (in fact at most $|C|$) have their origin in A. Hence $\widehat{C} := [A, \widehat{H} - A]_{\widehat{H}}$ is a finite cut of $\widehat{H}$ because $\widehat{C}$ consists of C and all edges joining u to a vertex in A which is the origin of some ray in $\mathcal{U}$. By the bond-faithfulness of $\widehat{\Delta}$, $\widehat{C}$ is also a cut of $\widehat{G}$. Therefore $\widehat{C} = [B, \widehat{G} - B]_{\widehat{G}}$ for some $B \subseteq \widehat{G}$ and the notation may be so chosen that $A \subseteq B$ and $\widehat{H} - A \subseteq \widehat{G} - B$. Then it is easy to see that $A = B \cap H$, whence

$$[A, H - A]_H = C = [B - u, G - B]_G.$$

Hence C is also an odd cut of G, which implies that there exist two rays $Q, Q' \in \mathcal{U}$ which respectively have a tail in B and $G - B$. Since u is the origin of both $\widehat{Q}$ and $\widehat{Q}'$, $\widehat{C}$ ($= [B, \widehat{G} - B]_{\widehat{G}}$) contains an odd number of edges (hence at least one) of either $\widehat{Q}$ or $\widehat{Q}'$, according as u is a vertex of $\widehat{G} - B$ or B. By the definition of $\widehat{\Delta}$, this particular ray is therefore contained in $\widehat{H}$ because $\widehat{C}$ is a cut of $\widehat{H}$ and $\widehat{H} \in \widehat{\Delta}$. Since $A = B \cap H$, the case where $u \in V(\widehat{G} - B)$ and $\widehat{Q} \subseteq \widehat{H}$ gives rise to a contradiction because Q ($= \widehat{Q} - u$), having a tail in B, will have a tail in A. For the case where $u \in V(B)$ and $\widehat{Q}' \subseteq \widehat{H}$, the vertex u, being the origin of $\widehat{Q}'$ is a vertex of $\widehat{H}$, a contradiction to $u \notin V(A)$ and $\widehat{H} - A \subseteq \widehat{G} - B$. □

In view of Theorem 6.5, Proposition 6.4 is also true without the assumption of countability. In particular, if $\mathcal{R}$ is the set of all the non-dominated ends, we have the following.

Theorem 6.9 *A graph has a decomposition into non-dominated circuits if and only if it has enough non-dominated rays.*

Proof The necessity is evident and the sufficiency is a consequence of Proposition 6.4 and Theorem 6.5. □

Hence the property of having enough non-dominated rays is a sufficient condition for a graph to admit a circuit decomposition. Thus for any class of graphs in which all rays are non-dominated, the property of having enough rays is a necessary and sufficient condition. One such class is formed by the block-locally-finite graphs (i.e. the graphs all of whose blocks are locally finite). These graphs are interesting because they are a generalization of trees and of locally finite graphs. Indeed, they have the following characterization.

Lemma 6.10 *A graph is block-locally-finite if and only if it contains no dominated ray and no pair of infinitely edge-connected vertices.*

Proof The necessity is obvious and the sufficiency a consequence of Proposition 2.11. □

Another family of graphs for which having enough rays is a necessary and sufficient condition for circuit-decomposability, are those that have at most one vertex of infinite degree, see Proposition 9.12.

7 Peripheral regions, odd-type vertices and circuit decomposition.

7.1 Peripheral regions

Intuitively speaking, a ray that goes through infinitely many "successive" odd cuts, as in Figure 5 is

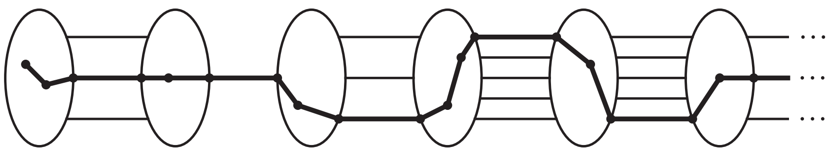

Figure 5:

always non-dominated. Hence in a graph which is not circuit-decomposable, and therefore, by Theorem 6.9, which does not have enough non-dominated rays, there must be odd regions that are "poor" in odd cuts. To make this idea precise, let us first define the following type of region:

Definition 7.1 An odd region of a graph G is *peripheral* if it contains no odd cut of G.

The word peripheral has been chosen because such regions can be visualized as lying on the "periphery" of the drawing of the graph. Note that a peripheral region A may contain a subgraph B such that $[B, \overline{B}]_G$ is an odd cut, but if this is so, then $[A, \overline{A}]_G \cap [B, \overline{B}]_G \neq \emptyset$ (see Figure 6). Moreover, if B is a

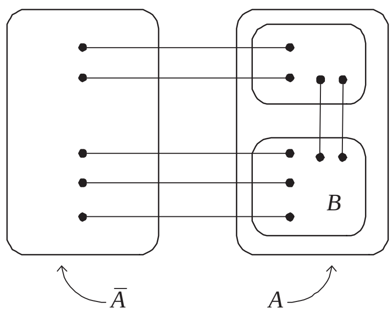

Figure 6:

region (i.e. connected), then it is automatically peripheral.

Proposition 7.2 *Every odd region of G contains a peripheral region of G or an ϵ-stratifying sequence of odd regions of G.*

Proof Suppose that G has an odd region A_0 which contains no peripheral region. Since A_0 itself is not peripheral, there exists an odd region $A_1 \subseteq A_0$ of G such that $[A_1, \overline{A_1}]_G \subseteq E(A_0)$ (and hence such that $A_1 \subseteq A_0 - \mathrm{Bdry}(A_0)$). Repeating this argument ad infinitum, one can construct the desired ϵ-stratifying sequence $(A_i)_{i\in\omega}$. □

Definition 7.3 A region of a graph is *obstructive* if all of its rays are dominated. (In particular a region which contains no rays is obstructive.)

The next three results will show that obstructive peripheral regions are the only parts of the graph where one may expect to encounter serious difficulties in connection with circuit decompositions. Recall that if a graph does not have enough non-dominated rays, then by definition it has an odd region all of whose rays are dominated.

Proposition 7.4 *Every obstructive odd region contains a peripheral region.*

Proof Proposition 7.2, Proposition 2.17 (ii) and Remark 2.16 □

Proposition 7.4 and Theorem 6.9 together say that a graph having no peripheral region is circuit-decomposable. The next theorem points out, in a more specific way, the link that exists between the structure of the peripheral regions and the existence or non-existence of a circuit decomposition of the graph.

Theorem 7.5 *Let G be a graph. Then the following statements are equivalent.*

(i) *G is circuit-decomposable;*

(ii) *every peripheral region of G is locally circuit-decomposable;*

(iii) *every obstructive peripheral region is locally circuit-decomposable in G;*

(iv) *among all maximal families $(A_i)_{i\in I}$ of vertex-disjoint obstructive peripheral regions, there is one which consists of locally circuit-decomposable regions.*

Proof (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident.

(iv) $\Rightarrow$ (i). First observe that without loss of generality we may suppose G to be connected and eulerian because for any vertex x of odd degree, $\{x\}$ is a rayless peripheral region that is not locally circuit-decomposable in G, and neither is any peripheral region containing x. Let $(A_i)_{i\in I}$ be a maximal family of vertex-disjoint obstructive peripheral regions and suppose that each A_i is locally circuit-decomposable. For each $i \in I$, let L_i be a finite connected subgraph of A_i that contains all the vertices in $\mathrm{Bdry}_G(A_i)$, and Δ_i be a decomposition of $A_i \cup [A_i, \overline{A_i}]_G$ into circuits, $\overline{A_i}$-rays and $\overline{A}_i$-paths. Let K_i be the subgraph of A_i that is the union of all the fragments of Δ_i which are edge-disjoint from $L_i \cup [A_i, \overline{A_i}]_G$. Note that K_i is circuit-decomposable, and that $G\backslash K_i$ is still eulerian. Since the A_i's are vertex-disjoint, so are the K_i's; hence $H := G\backslash(\bigcup_{i\in I} K_i)$ is eulerian. We claim that H has enough non-dominated rays. Note that if this claim is true then we are done, since by Theorem 6.9, H is circuit-decomposable, and hence, the K_i's being pairwise disjoint, $(K_i)_{i\in I} \cup \{H\}$ is a decomposition of G into circuit-decomposable graphs.

To prove the claim, let us suppose by way of contradiction that H does not have enough non-dominated rays. By Proposition 7.4, there is a peripheral region B of H that contains no non-dominated rays. Now put

$$B_i := A_i\backslash K_i \ \ (= A_i \cap H), \ \ i \in I.$$

Since $B_i \cup [A_i, \overline{A_i}]_G$ is the Δ_i-shadow of $L_i \cup [A_i, \overline{A_i}]_G$ for any $i \in I$, we have that $[B_i, \overline{B_i}]_H = [A_i, \overline{A_i}]_G$ and that B_i is infinite, locally finite and connected for any $i \in I$. There are now two cases to consider.

Case 1. $B \cap (\bigcup_{i\in I} B_i)$ is infinite. In that case, there must exist an $i_0 \in I$ such that $B_{i_0} \cap B$ is infinite because otherwise, each B_i being infinite and connected, an infinite number of them would have an edge in the finite cut $[B, \overline{B}]_H$, contradicting the fact that the B_i's are pairwise disjoint. Since B_{i_0} is connected and $[B, \overline{B}]_H$ is finite, $B_{i_0} \cap B$ contains a finite number of components. One of these components is therefore infinite. This particular component contains a ray (say R) because it is an infinite connected subgraph of the locally finite graph B_{i_0}. Being contained in B_{i_0}, which is a locally finite region of H, R is therfore not dominated in H, a contradiction.

Case 2. $B \cap (\bigcup_{i\in I} B_i)$ is finite (possibly empty). Let $D := B \cap (\bigcup_{i\in I} B_i)$. Since the B_i's are pairwise disjoint locally finite regions of the eulerian graph H, every vertex of D is of even degree in H. This implies that $\left|[D, \overline{D}]_H\right|$ is even because it is equal to $\sum_{x\in V(D)} \deg_H(x) - 2\left|E(D)\right|$. Hence, by Lemma 2.2, $[B - D, \overline{B - D}]_H$ is an odd cut. Being contained in B, $B - D$ must by Proposition 7.4 contain an obstructive peripheral region C of H. However, C is disjoint from each $A_i, i \in I$, implying that it is an induced subgraph of G, and hence an obstructive peripheral region of G, a contradiction to the maximality of the family $(A_i)_{i\in I}$. □

7.2 The parity type of vertices

Observe that in a finite and also in an infinite locally finite graph G, a peripheral region A always contains an odd number of vertices whose degree in G is odd, and these vertices must belong to $\mathrm{Bdry}_G(A)$. In general, this is not true because of the vertices of infinite degree, which in a sense may be considered of both odd and even degree. This double status is clearly the basic reason why circuit decompositions become so difficult to study once we go beyond the locally finite case. The following is a generalization to vertices of infinite degree of the parity property that exists for vertices of finite degree.

Definition 7.6 A vertex x of a graph G is said to be of *even type* in G if every odd region of G that contains x also contains an even region which still contains x. A vertex which is not of even type is said to be of *odd type*.

Note that any two infinitely edge-connected vertices are always of the same type, hence we can define an *even-type class* (resp. *odd-type class*) as an ω-class whose members are of even type (resp. odd type).

It is easy to see that vertices of even or odd degree are respectively of even type or odd type. Moreover, as shown by the results of the rest of section 7, there is a close connection between odd-type vertices and peripheral regions (the "problematic" parts of the graph from the point of view of circuit-decomposability) and between even-type vertices and locally cycle-decomposable regions (the "nice" parts). Moreover, since by Nash-Williams's Theorem locally cycle-decomposable regions cannot contain odd cuts, they are in some sense the even counterparts of the peripheral regions.

Proposition 7.7 *An odd-type vertex always belongs to some peripheral region of the graph.*

Proof Without loss of generality suppose that G is connected and by way of contradiction, let x be an odd-type vertex of G that does not belong to any peripheral region of G. By the definition of odd type, G has an odd region A that contains x and such that no even region containing x is contained in A. Now observe that if there exists an odd cut $[C, \overline{C}]_G \subseteq E(A)$ such that $A \cap C$ is connected and contains all the vertices of $\mathrm{Bdry}_G(A)$, then $A \cap C$ is an even region because in such case

$$[A \cap C, \overline{A \cap C}]_G = [A, \overline{A}]_G \,\dot\cup\, [C, \overline{C}]_G.$$

Thus to reach a contradiction, let us construct such a cut. Denote by X the ω-class of x in G, and choose a finite tree $T \subseteq A$ that contains x and every vertex of $\mathrm{Bdry}_G(A)$. Let $B \subseteq A$ be a region that contains x but that is vertex-disjoint from $T - X$; by Lemma 2.3, such a region exists because T is finite and no vertex of $T - X$ is infinitely edge-connected to x in G. Note that by assumption, B must be an odd region but not a peripheral one. Thus there exists an odd cut $[C, \overline{C}]_G \subseteq E(B)$ with $x \in V(C)$. Since a cut is an edge-disjoint union of bonds, we may suppose without loss of generality that $[C, \overline{C}]_G$ is a bond and therefore that C is connected. T is contained in C because the only edges of T that belong to B are those that have both endpoints in X, and therefore must belong to C. Hence since T is connected and contained in A, and since

$$\{x\} \cup \mathrm{Bdry}_G(A) \subseteq V(T),$$

$A \cap C$ is likewise connected and contains x and the vertices of $\mathrm{Bdry}_G(A)$. This completes the proof. □

Corollary 7.8 *The vertices of a cycle-decomposable graph are all of even-type.*

The next result shows that in appropriate circumstances parity type is preserved under the operation of taking quotients.

Lemma 7.9 *Let A be any induced subgraph of G such that $[A, \overline{A}]_G$ is finite, and let $x \in V(A)$. Then x is of even type in G if and only if it is of even type in $G/\overline{A}$.*

Proof ($\Rightarrow$:) By way of contradiction suppose that x is of even-type in G and an odd region B of $G/\overline{A}$ contains x but contains no even region of $G/\overline{A}$ which contains x. Without loss of generality we may suppose that $B \subseteq A$ (i.e., $q_{\overline{A}} \notin V(B)$) because otherwise, $\deg_{G/\overline{A}}(q_{\overline{A}})$ being finite, the component of $B - q_{\overline{A}}$ that contains the vertex x will be a region of $G/\overline{A}$ contained in B, and hence an odd region that contains no even region that contains x. Thus B is also an odd region of G that contains x. Now since x is of even type in G, there exists an even region B' of G such that $x \in B' \subseteq B$, which gives rise to a contradiction because B' is also an even region of $G/\overline{A}$.

($\Leftarrow$:) Again by way of contradiction suppose that x is of even-type in $G/\overline{A}$ and B is an odd region of G which contains x but whose even subregions do not contain x. This implies that the component K of $B \cap A$ that contains x is an odd region of G, and hence clearly an odd region of $G/\overline{A}$. Now we obtain our contradiction by taking any even region of $G/\overline{A}$ that contains x and is contained in K, any such even region of $G/\overline{A}$ being also an even region of G. $\square$

We now establish the property — already mentioned in the introduction— that the parity type of a vertex does not change if we remove a finite eulerian subgraph. In fact the removal of an arbitrary eulerian subgraph H of G may only change the parity type of the vertices that ϵ-dominate H in G, where the concept of ϵ-domination of a subgraph (analogous to the concept of domination introduced in Section 5) is defined as follows: a vertex x is said to *ϵ-dominate a subgraph H of G* if there is an infinite family of edge-disjoint xH-paths having different end-vertices in H, or equivalently, if every region of G that contains x, contains infinitely many vertices of H.

Proposition 7.10 *Let H be any eulerian subgraph of G and x any vertex that does not ϵ-dominate H in G. Then x is of even type in G if and only if it is of even type in $G \backslash H$.*

Proof *Case 1.* H is finite.
($\Rightarrow$:) Let A be any odd region of $G\backslash H$ that contains x, and let us show that A contains an even region that contains x. By Lemma 2.3, choose a region $B \subseteq A$ such that $V(B) \cap V(H)$ is contained in the ω-class of x. If B is even we are done. So suppose that B is odd (in $G\backslash H$). Let B^+ be the subgraph of G induced by $V(B)$ (i.e., $B^+\backslash H = B$). Since H is eulerian and finite, B^+ is an odd region of G. Consequently, there exists an even region C of G that is contained in B^+ and contains x. Clearly, $[C\backslash H, \overline{C\backslash H}]_{G\backslash H}$ is an even cut. Moreover, C being connected, so also is $C\backslash H$ because otherwise there exists an edge $e \in E(C) \cap E(H)$ whose two incident vertices (x_1, x_2, say) belong to different components of $C\backslash H$. But this is a contradiction because, being in the same ω-class of G, x_1 and x_2 cannot be separated by the

removal of the edges constituting the finite set $E(H) \cup [C, \overline{C}]_G$. Thus $C \backslash H$ is the desired even region of $G \backslash H$.

($\Leftarrow$:) Let A be any odd region of G, and B the component of $A \backslash H$ that contains x. Since $[A \backslash H, \overline{A \backslash H}]_{G \backslash H}$ is finite, B is therefore a region of $G \backslash H$. Let D be an even region of $G \backslash H$ that is contained in B and contains x (if B is already an even region, put $D := B$). Then the subgraph of G induced by $V(D)$ is an even region of G contained in A that contains x.

Case 2. H is infinite. Since x does not ϵ-dominate H, there is a region A of G that contains x and such that $A \cap H$ is finite. It follows from that finiteness that

$$\deg_{H/\overline{A}}(q_{\overline{A}}) \;=\; \sum_{x \in V(A \cap H)} \deg_H(x) \;-\; 2\Big|E(A \cap H)\Big|,$$

and hence that $H/\overline{A}$ is a finite eulerian subgraph of $G/\overline{A}$ (possibly a single vertex). Moreover, $[A \backslash H, \overline{A \backslash H}]_{G \backslash H}$ is a finite cut. Hence we obtain that x is of the same parity type in G, in $G/\overline{A}$, in $(G/\overline{A}) \backslash (H/\overline{A})$ $(= (G \backslash H)/(\overline{A \backslash H}))$, and in $G \backslash H$, by applying Lemma 7.9 to G and A, the present proposition (Case 1) to $G/\overline{A}$ and $H/\overline{A}$, and Lemma 7.9 to $G \backslash H$ and $A \backslash H$. $\square$

7.3 Vertices of even type and locally cycle-decomposable regions

Lemma 7.11 *Let X be an ω-class of G contained in some peripheral region A of G, and let $B \subseteq A$ be a region that contains X but no vertex of* $\mathrm{Bdry}_G(A) - X$*. Then the following statements are equivalent.*

(1) *X is of even type;*

(2) *B is an even region of G;*

(3) *B is locally cycle-decomposable in G.*

Note that since $\mathrm{Bdry}_G(A)$ is finite and X is an ω-class of G, Lemma 2.3 implies that there always exists a region $B \subseteq A$ such that $X \subseteq V(B)$ and $V(B) \cap \mathrm{Bdry}_G(A) \subseteq X$.

Proof $(3) \Rightarrow (1)$. Observe that, since $G/\overline{B}$ is cycle-decomposable, by Nash-Williams's Theorem X must be of even type in $G/\overline{B}$ and hence, by Lemma 7.9, of even type in G.

$(2) \Rightarrow (3)$. By way of contradiction, suppose that B is an even region that is not locally cycle-decomposable. By Nash-Williams's Theorem, there is an odd cut $[C, (G/\overline{B}) - C]_{G/\overline{B}}$ of $G/\overline{B}$ such that $X \subseteq V(C)$. Note that we may suppose without loss of generality that $q_{\overline{B}} \in V(C)$, because otherwise consider the cut induced by $C \cup \{q_{\overline{B}}\}$ instead of C: by Lemma 2.2, the new cut will then also be odd since $q_{\overline{B}}$ is a vertex of even degree in $G/\overline{B}$. Since $q_{\overline{B}} \in V(C)$, it follows that $F := (G/\overline{B}) - C$ is a subgraph of G. Moreover $[F, \overline{F}]_G = [C, G/\overline{B} - C]_{G/\overline{B}}$ and so $[F, \overline{F}]_G$ is an odd cut of G that is contained in $B \cup [B, \overline{B}]_G$, and hence in $A \cup [A, \overline{A}]_G$. Moreover, since $V(B) \cap \mathrm{Bdry}_G(A) \subseteq X$ and

$V(F) \cap X = \emptyset$, no edge of $[F, \overline{F}]_G \cap [B, \overline{B}]_G$ belongs to $[A, \overline{A}]_G$. Thus $[F, \overline{F}]_G$ is totally contained in A, contradicting the fact that A is peripheral in G.

$(1) \Rightarrow (2)$. By way of contradiction suppose that B is an odd region of G. Since X is assumed to be of even type, there exists an even region C of G that contains X and is contained in B. Put $D := B - C$. As one can easily see in Figure 7, this implies that

$$\left|[D, \overline{D}]_G\right| = \left|[B, \overline{B}]_G\right| - \left|[C, \overline{B}]_G\right| + \left|[C, \overline{C}]_G\right| - \left|[C, \overline{B}]_G\right|.$$

Since $\left|[B, \overline{B}]_G\right|$ and $\left|[C, \overline{C}]_G\right|$ are of different parity, $[D, \overline{D}]_G$ must be an odd cut. However, since C contains X, and B contains no element of $\mathrm{Bdry}_G(A) - X$, it follows that A contains the odd cut $[D, \overline{D}]_G$, a contradiction since A is peripheral. $\square$

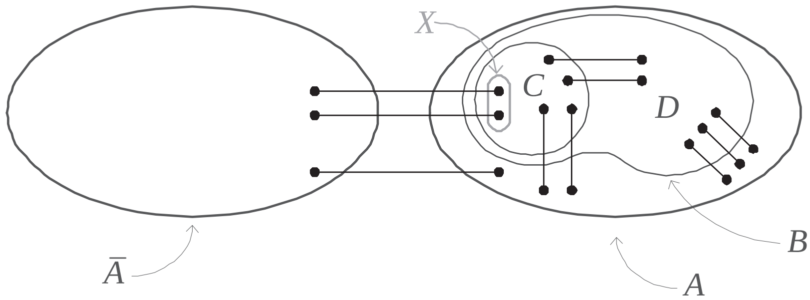

Figure 7:

The preceding result establishes a certain link between even-type vertices and locally cycle-decomposable regions of the graph. This link comes out in an even stronger way in the following proposition which is a generalization to the infinite case of a basic property of the vertices of even degree in finite graphs.

Proposition 7.12 *Let B be a region of a graph G. Then*

$$B \text{ is locally cycle-decomposable} \Rightarrow \text{each vertex of } B \text{ is of even type in } G.$$

Moreover, if B is contained in some peripheral region of G, the converse is also true.

Proof If B is locally cycle-decomposable then, by Nash-Williams's Theorem and since $G/\overline{B}$ is cycle-decomposable, each vertex of B must be of even type in $G/\overline{B}$, and hence by Lemma 7.9, of even type in G. Now, suppose that B is composed of even type vertices of G only, that it is contained in some peripheral region A, and let us show that it is locally cycle-decomposable. By way of contradiction, suppose that $G/\overline{B}$ is not cycle-decomposable, and by Nash-Williams's Theorem, let C be an odd region of $G/\overline{B}$ such that $q_{\overline{B}} \notin V(C)$. Then C is also an odd region of G contained in B, and hence in A. Since A is peripheral, $[C, \overline{C}]_G \nsubseteq E(A)$, which implies that $C \cap \mathrm{Bdry}_G(A)$ is non empty. Since moreover it is finite, without loss of generality, we may suppose C has been chosen such that $C \cap \mathrm{Bdry}_G(A)$ has smallest possible cardinality, and let $x \in C \cap \mathrm{Bdry}_G(A)$. Since x is of even type in G and lies in C, it is contained in some even region $D_x \subseteq C$ of G. By Lemma 2.2, $[C - D_x, \overline{C - D_x}]_G$ is an odd cut, and since $C - D_x$ is a union of disjoint regions, there therefore exists an odd region C_x of G that is a component of $C - D_x$. Clearly, C_x is an odd region of $G/\overline{B}$ such that $q_{\overline{B}} \notin V(C)$, a contradiction to the minimality assumption because

$$C_x \cap \mathrm{Bdry}_G(A) \subseteq C - D_x \cap \mathrm{Bdry}_G(A) \subsetneq C \cap \mathrm{Bdry}_G(A).$$

$\square$

7.4 Vertices of odd type and peripheral regions

The next proposition plays the same role for odd-type vertices as Proposition 7.12 does for the vertices of even type. In other words, it shows that some properties that are satisfied by the vertices of odd degree in a finite graph are also satisfied by the vertices of odd type in an arbitrary graph, and also points out a link that exists between odd-type vertices and peripheral regions.

Proposition 7.13 *Let G be a graph. Then*

(i) *every peripheral region A contains an odd number of odd-type classes and each such class meets* $\mathrm{Bdry}_G(A)$*;*

(ii) *for every odd-type class O and every region A that contains O there exists a peripheral region* $B \subseteq A$ *such that O is the only odd-type class of G to be contained in B.*

The proof of this proposition is based on the following lemma.

Lemma 7.14 *Let A be a peripheral region of G and* $(P_i)_{i \in I}$ *be a maximal set of edge-disjoint paths in A such that the two endpoints of each* P_i *belong to* $\mathrm{Bdry}_G(A)$ *but are in different* ω*-classes of G. Then* $H := (\bigcup_{i \in I} P_i) \cup [A, \overline{A}]_G$ *is finite,* $A \backslash H$ *is cycle-decomposable, and for each* ω*-class X contained in A we have:*

$$X \text{ is of odd type in } G \iff \begin{array}{l} X \text{ contains an odd number of vertices} \\ \text{whose degree in } H \text{ is odd.} \end{array}$$

Note that in the case where $\mathrm{Bdry}_G(A)$ is contained in a single ω-class of G (in particular when $[A, \overline{A}]_G$ is a bridge), Lemma 7.14 asserts that A is cycle-decomposable. (See Figure 9 at the end of the paper for some examples.)

Proof Clearly I is finite because so is $\mathrm{Bdry}_G(A)$ and because between two different ω-classes of G there is only a finite number of edge-disjoint paths. Hence H is finite.

Let K be the subgraph of G induced by $V(\overline{A}) \cup \mathrm{Bdry}_G(A)$. Since A is peripheral in G, G/K has no odd cut and hence, by Nash-Williams's Theorem, is cycle-decomposable. Moreover, since I is finite and P_i/K a cycle or an edge-disjoint union of several cycles, for any $i \in I$, $(G - \bigcup_{i\in I} P_i)/K$ is also cycle-decomposable. Assume $A \backslash H$ is not cycle-decomposable. Since $(A\backslash H)/K = (G - \bigcup_{i\in I} P_i)/K$, there exist two vertices of $\mathrm{Bdry}_G(A)$ that belong to the same component of $A\backslash H$ but which are separated in that component by an odd (and hence finite) cut, contradicting the maximality of the family $(P_i)_{i\in I}$. Thus $A\backslash H$ is cycle-decomposable.

Now let X be any ω-class of G that is contained in A, and let $B \subseteq A$ be any region of G that contains X but no vertex of $\mathrm{Bdry}_G(A) - X$. As noted after the statement of Lemma 7.11, such a B exists. Moreover, since $A\backslash H$ contains no odd cut, B is an odd region if and only if X contains an odd number of vertices of odd degree in H. Thus by Lemma 7.11 we are done. $\square$

Proof of Proposition 7.13 (i) Take any finite subgraph H as defined in Lemma 7.14, and put $G_1 := (G\backslash A) \cup H$. Since H is finite and $[A \cap H, G_1 - (A \cap H)]_{G_1}$ is an odd cut of G_1, $A \cap H$ contains an odd number of vertices whose degree in G_1 is odd. Since vertices of $A \cap H$ have the same degree in H as in G_1, it follows that $|V(A) \cap V_{odd}(H)|$ is odd where $V_{odd}(H)$ is the set of vertices whose degree in H is odd. Therefore there are an odd number of ω-classes $X \subseteq V(A)$ such that $|X \cap V_{odd}(H)|$ is odd. By Lemma 7.14, these ω-classes are precisely the odd-type classes contained in A. Moreover, if X is one of these ω-classes, then $\emptyset \neq X \cap V_{odd}(H) \subseteq \mathrm{Bdry}_G(A)$ because vertices in $V(A \cap H) - \mathrm{Bdry}_G(A)$ have even degree in H. This proves part (i) of Proposition 7.13.

(ii) By Proposition 7.7, there is a peripheral region C of G that contains O. Since, by Lemma 2.2, $[C \cap A, \overline{C \cap A}]_G$ is finite, there exists a region D of G such that $O \subseteq V(D) \subseteq V(C \cap A)$. By Lemma 2.3 (with A, Y, x replaced by D, $\mathrm{Bdry}_G(C) - O$, and an arbitrary element of O, respectively), there exists a region B of G such that $B \subseteq D$, $O \subseteq V(B)$ and

$$B \cap \mathrm{Bdry}_G(C) \subseteq O. \tag{1}$$

Since B is contained in D, it is contained in the peripheral region C. Hence, it follows from Lemma 7.11 (with X, A, B replaced by O, C, B, respectively) that B is an odd and therefore peripheral region of G. Moreover, it follows from (i) (with A replaced by C), from equation (1) and from the fact that an ω-class is either disjoint from $V(B)$ or contained in it, that B contains no odd-type class other than O. Thus we are done. $\square$

Proposition 7.13 says in particular that a graph which has no odd-type vertex does not have any peripheral region. Thus by Theorem 7.5 we have the following corollary of Proposition 7.13.

Corollary 7.15 *A graph all of whose vertices are of even type is circuit-decomposable.*

For vertex transitive graphs, Corollary 7.15 has the following interesting consequence.

Corollary 7.16 *A vertex transitive graph is circuit-decomposable if and only if it is eulerian.*

Proof Let G be an eulerian vertex transitive graph. By Corollary 7.15, we only have to show that every vertex of G is of even type. Assume the contrary. Then since G is vertex transitive, every vertex of G is of odd type. Let A be a peripheral region that contains exactly one odd-type class of G. Denote this class by O and observe that $O = V(A)$. This implies that O must be an infinite set and therefore that $V(A) - \mathrm{Bdry}_G(A) \neq \emptyset$. Moreover, since O is an ω-class, no subgraph of A, except A itself, can be a region of G. Thus no automorphism of G can map a vertex of $\mathrm{Bdry}_G(A)$ to a vertex of $V(A) - \mathrm{Bdry}_G(A)$, a contradiction. □

7.5 Peripheral regions containing exactly one odd-type class

The peripheral regions of a graph that contain exactly one odd-type class are of particular interest because, as is shown by the following results, the circuit-decomposability of the whole graph is equivalent to the local circuit-decomposability of these regions, and because they have the surprising property that, regardless of whether they are locally circuit-decomposable or not, they are always locally cycle-decomposable *to within a single edge*.

First note that by Proposition 7.13 (ii) and the definition of an odd-type class (Definition 7.6), we have the following remark:

Remark 7.17 Every odd-type class of a graph is contained in some peripheral region that contains no other ω-class of odd type.

This implies the following result.

Proposition 7.18 *A graph G is circuit-decomposable if and only if every obstructive peripheral region of G that contains exactly one odd-type class is locally circuit-decomposable.*

Proof Necessity follows from Theorem 7.5. For the sufficiency it follows from Proposition 7.13 (ii) that any maximal family of vertex-disjoint obstructive peripheral regions containing each at most one odd-type class, is also maximal when considered simply as a family of vertex-disjoint peripheral regions having only dominated rays. Thus the result follows by the implication (iv) ⇒ (i) of Theorem 7.5. □

Proposition 7.19 *Let A be a peripheral region of G containing exactly one odd-type class O. Then $(G\backslash e)/\overline{A}$ ($=(G/\overline{A})\backslash e$) is cycle-decomposable for any $e \in [O,\overline{A}]_G$.*

Proof By Proposition 7.13 (i), there exists an edge $e \in [O,\overline{A}]_G$. Let us show that $G_1 := (G\backslash e)/\overline{A}$ is cycle-decomposable. Suppose the contrary. Then by Nash-Williams's Theorem there exists an odd cut $[C,\overline{C}]_{G_1}$ such that $O \subseteq V(C)$. Since $\overline{A}$ is of even degree in G_1, we may suppose without loss of generality that $\overline{A} \in V(C)$, because otherwise, consider the cut induced by $C \cup \{\overline{A}\}$ instead of C (by Lemma 2.2 both are odd cuts). Hence, $V(\overline{A}) \cup O \subseteq V(C)$, which implies that $[C,\overline{C}]_{G_1}$ is also an odd cut of G and that $\overline{C}$ is contained in the peripheral region A. This implies that one of the components of $\overline{C}$ is also a peripheral region of G. Thus, by Proposition 7.13(i), $\overline{C}$ contains an odd type class of G, a contradiction to the fact that O is the only odd type class of G contained in A. □

The next result shows that peripheral regions containing exactly one odd-type class have a kind of extremal property with respect to all the regions of the graph.

Theorem 7.20 *Let A be a peripheral region of G that contains a unique odd-type class O. Then any region B contained in A is either a locally cycle-decomposable (and hence even) region of G that is disjoint from O or a peripheral (and hence odd) region that contains O.*

Proof Let $B \subseteq A$ be a region of G and note that O is either disjoint from $V(B)$ or contained in it. In the first case, by Proposition 7.12, B must be a locally cycle-decomposable even region. In the second case, suppose by way of contradiction that B is not a peripheral region. Since $B \subseteq A$, and A is peripheral, B does not contain odd cuts, and hence must be an even region because otherwise it would be peripheral. By Lemma 2.2, this implies that $[A-B,\overline{A-B}]_G$ is an odd cut of G, and therefore that at least one component C of $A-B$ is an odd region that contains no odd-type class of G. Since any odd region contained in a peripheral one is also peripheral, it follows that C is peripheral, and this contradicts Proposition 7.13 (i). □

Corollary 7.21 *Let A be a peripheral region having exactly one odd-type class. If A is locally circuit-decomposable, then one of the following holds:*

(i) *$G/\overline{A}$ has a decomposition into cycles and exactly one $q_{\overline{A}}$-ray.*

(ii) *A contains an even region B of G such that $G/\overline{B}$ has a decomposition into cycles and exactly two $q_{\overline{B}}$-rays.*

Proof Let O be the unique odd-type class of G contained in A and Δ any decomposition of $G/\overline{A}$ into circuits and $q_{\overline{A}}$-rays. Since $q_{\overline{A}}$ is a vertex of odd degree in $G/\overline{A}$, we may suppose without loss of generality that Δ contains exactly one $q_{\overline{A}}$-ray. Now we have two cases to consider.

Case 1. Some infinite fragment of Δ has a tail that is not ϵ-dominated in G by any vertex of O. Then there exists a region $B \subseteq A$ of G which is disjoint from O and contains a tail of some fragment of Δ. Then by Proposition 7.12, B must be a locally cycle-decomposable even region, and it is easy to see that in addition, $G/\overline{B}$ satisfies the conditions of Lemma 3.8. Thus $G/\overline{B}$ has a decomposition into cycles and exactly two $q_{\overline{B}}$-rays.

Case 2. Each tail of each infinite fragment of Δ is ϵ-dominated in G by some (and hence every) vertex of O. By Proposition 7.13 (i), there exists an edge $e \in [O, \overline{A}]_G$. Since A is connected, there is a $q_{\overline{A}}$-ray R in $G/\overline{A}$ such that e is the first edge of R and R shares a tail with the only ray of Δ. If we remove R from its Δ-shadow, the resulting subgraph is eulerian and locally finite and so is decomposable into circuits. Therefore $(G/\overline{A})\backslash R$ has a circuit decomposition, and so $G/\overline{A}$ has a decomposition into circuits and exactly one $q_{\overline{A}}$-ray, namely R. Denote by u the second vertex of R. Since $(G\backslash e)/\overline{A}$ admits a decomposition into circuits and exactly one u-ray, by Proposition 7.19 it also admits a cycle decomposition and thus, by Proposition 3.7 for $n = 1$, admits a decomposition into cycles and exactly one u-ray R'. Since $\{e\} \cup E(R')$ is the set of edges of either a $q_{\overline{A}}$-ray or the edge-disjoint union of a $q_{\overline{A}}$-ray and a cycle, $G/\overline{A}$ therefore admits a decomposition into cycles and exactly one $q_{\overline{A}}$-ray as claimed. □

8 Removable subgraphs

In this section, we look for types of subgraphs which can be removed from the original graph without changing its circuit-decomposability or circuit-indecomposability. Hence the results of this section are generalizations to the situation of circuit-decomposability of the fact that removing any finite eulerian subgraph from a (finite or infinite) graph G does not change the *cycle*-decomposability of G.

Proposition 8.1 *Let H be a non-dominated eulerian subgraph of G. If G is circuit-decomposable, then so is $G\backslash H$, and conversely.*

Proof Since H is locally finite by Corollary 5.5 and eulerian, it is circuit-decomposable by Theorem 2.24. Therefore G is circuit-decomposable if $G\backslash H$ is circuit-decomposable.

To prove the converse, assume that G has a circuit decomposition Δ. Let K be the Δ-shadow of H. Since $G\backslash K$ and H are circuit-decomposable, and $K \supseteq H$, it is sufficient to show that $K\backslash H$ also has such a decomposition. Being circuit-decomposable, K is eulerian, hence so is $K\backslash H$ because H is locally finite and eulerian. Let $\theta = (\theta_x)_{x\in V(G)}$ be the transition system of G induced by Δ. For $x \in V(K\backslash H)$, let θ'_x be the set of all pairs $\{e, e'\} \in \theta_x$ such that $e, e' \in E(K\backslash H)$. Since H is locally finite and eulerian, the number of edges of $K\backslash H$ incident with x but not belonging to any pair of θ'_x is even. Therefore θ'_x can be extended to a (full) transition system θ''_x at x. Let $\theta'' = (\theta''_x)_{x\in V(K\backslash H)}$. By Theorem 2.26, θ'' defines a decomposition Θ'' of $K\backslash H$ into edge-traceable fragments.

It will suffice to prove that each fragment of Θ'' is circuit-decomposable. Therefore suppose, by way of contradiction, that a fragment $M \in \Theta''$ is not circuit-decomposable. Then, by Theorem 6.9, there is an odd region A of M such that every ray of A is dominated in M. Since A is a subgraph of a edge-traceable graph, it is countable or finite, and hence by Theorem 2.10 has a rayless spanning tree T. Hence

$$V(A) \cap V(H) \text{ is finite,} \tag{2}$$

because, by Lemma 5.3, so is $V(T) \cap V(H)$. Note, however, that A must be infinite because otherwise the quotient graph $M/(M-A)$ is a finite graph with exactly one vertex of odd degree, viz. q_{M-A}. Hence A must contain a vertex u of infinite degree, because otherwise A is infinite and locally finite and therefore contains a ray, and clearly no such ray can be dominated in A.

Now let $\langle \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \rangle$ be the 2-way infinite trail of M induced by θ''. Let $U := \{i; x_i = u\}$. Note that U is infinite and since $[A, M-A]_M$ is odd, it must either contain a least or a greatest element; without loss of generality suppose it has a smallest one, and let $U = \{i_0, i_1, \ldots\}$ with $i_n < i_{n+1}$. Let n_A be a non-negative integer such that $x_i \in V(A)$ for any $i \geq i_{n_A}$. Observe that for each $n \geq 0$, the sub-trail $S_n := \langle x_{i_n}, x_{i_n+1}, \ldots, x_{i_{n+1}}, x_{i_{n+1}+1} \rangle$ contains three different edges that are incident with u. Since the transitions in θ arise from circuits (i.e. from 2-regular graphs), at least one of the θ''-transitions in S_n does not belong to θ. This means that there exists an index j_n with $i_n < j_n \leq i_{n+1}$ such that $e_n := [x_{j_n-1}, x_{j_n}]$ and $e'_n = [x_{j_n}, x_{j_n+1}]$ do not form a pair in $\theta_{x_{j_n}}$. Thus, for each $n \geq n_A$, x_{j_n} belongs to $V(A) \cap V(H)$, and both e_n and e'_n are paired in $\theta_{x_{j_n}}$ with edges of H. Since H is locally finite, it follows that $V(H) \cap V(A)$ is infinite, a contradiction to (2). $\square$

Corollary 8.2 *Let H be an eulerian subgraph of a circuit-decomposable graph G, and D the set of all vertices of G that dominate H. Then $G \backslash H$ has a decomposition into circuits, D-rays and D-paths.*

Proof Without loss of generality suppose that each vertex adjacent to a vertex of D is of degree 2 and the distance in G between any two vertices of D is at least 3. If this is not the case, subdivide each edge of G (and consequently of H) by two vertices. For each $e \in E(G)$ incident to a vertex of D, introduce a new ray R_e at the vertex of e that is not in D, such that R_e meets G in that vertex only and such that the R_e's are pairwise disjoint. Now let

$$\widehat{G} := (G - D) \cup \bigcup_{\substack{e \in E(G) \\ V(e) \cap D \neq \emptyset}} R_e$$

and

$$\widehat{H} := (H - D) \cup \bigcup_{\substack{e \in E(H) \\ V(e) \cap D \neq \emptyset}} R_e.$$

It is easy to see that $\widehat{H}$ is an eulerian non-dominated subgraph of $\widehat{G}$ and that $\widehat{G}$ is circuit-decomposable. Hence, by Proposition 8.1, $\widehat{G} \backslash \widehat{H}$ has a decomposition into circuits and since any such decomposition canonically induces a decomposition of $G\backslash H$ into circuits, D-rays and D-paths, we are done. □

Intuitively, the last result together with Corollary 7.15 says that if one wishes to remove some subgraph H of a circuit-decomposable graph G in such a way that the remaining graph is still circuit-decomposable, a possible class of candidates for H are those subgraphs which are not dominated in G by any vertex that is of odd type in $G\backslash H$. Pursuing this idea further we will extend Proposition 8.1 to the following class of subgraphs (see Theorem 8.6).

Definition 8.3 A subgraph H of a graph G is said to be *removable* if no vertex that dominates H in G is of odd type in $G\backslash H$.

An important situation where this concept is used is the case where $G\backslash H$ is cycle-decomposable. Since in this case $G\backslash H$ has no vertices of odd type (see Corollary 7.8), H is a removable subgraph of G.

As is the case for the parity-type of vertices (cf. Lemma 7.9), removability is preserved by the passage from a graph to some of its quotients:

Lemma 8.4 *Let H be a subgraph of G and A a region of G. Then*

$$H \text{ is removable in } G \Longrightarrow H/\overline{A} \text{ is removable in } G/\overline{A}.$$

Moreover, if A contains all the vertices of G that dominate H, the converse is also true.

Proof Straightforward from Lemma 7.9, the definition of removable graphs and the fact that since $[A, \overline{A}]_G$ is finite, the vertices that dominate $H/\overline{A}$ in $G/\overline{A}$ are exactly the vertices of A that dominate H in G. □

Definition 8.5 An *eulerian-type* graphs is a graph that contains no odd-type vertex.

The following result which is a generalization of Theorem 8.1, show that removable subgraphs are "generalized" non-dominated subgraphs.

Theorem 8.6 *Let H be a removable eulerian-type subgraph of G. If G is circuit-decomposable, then so is $G\backslash H$, and conversely.*

Proof Sufficiency. Since all vertices of H are of even type in H, by Corollary 7.15, H is circuit-decomposable and hence so is G.

Necessity. Let $K := G\backslash H$. We shall show that any obstructive peripheral region of K is locally circuit-decomposable. The circuit decomposability of K then follows from the implication (iii)⇒(i) of Theorem 7.5.

Let A be an obstructive peripheral region of K. Denote by D the set of all vertices that dominate H in G, and put $D_A := D \cap V(A)$. Let O be the set of all vertices that are of odd type in K, and let $O_A := O \cap V(A)$.

Note that because H is removable the two sets D and O are disjoint and hence so are D_A and O_A. We now claim that D_A and O_A can be separated by a finite cut, or, to be more specific:

Claim: *There exists an induced subgraph B of A such that $[B, A-B]_A$ is finite, $D_A \subseteq V(B)$ and $O_A \subseteq V(A-B)$.*

Assuming for the moment that this claim is true, let us show that it implies the result.

First note that since both $[A, \overline{A}]_K$ and $[B, A-B]_A$ are finite, so are $[B, K-B]_K$ and $[A-B, K-(A-B)]_K$. By Corrollary 8.2, K admits a decomposition Δ into circuits, D-rays and D-paths. Because of the finiteness of $[A-B, K-(A-B)]_K$ and the fact that $A-B$ is an induced subgraph of K, Δ may contain at most a finite number of fragments that meet $A-B$ without being included in it. Since $D \cap V(A-B) = \emptyset$, the fragments of Δ contained in $A-B$ are circuits, and hence their union is a circuit-decomposable subgraph L of $A-B$ such that every $x \in V(A-B)$ is of even degree in $K\backslash L$.

Now, put $\overline{B} := K - B$. Since $V(B) \cap O = \emptyset$, Lemma 7.9 implies that there is at most one odd-type vertex in $K/\overline{B}$, viz. $q_{\overline{B}}$. Hence, if the vertex $q_{\overline{B}}$ is not of odd type in $K/\overline{B}$, then by Corollary 7.15, $K/\overline{B}$ is circuit-decomposable. On the other hand, if $q_{\overline{B}}$ is of odd type, we attach a $q_{\overline{B}}$-ray to $K/\overline{B}$, obtaining a new graph with even type vertices only. By Corollary 7.15 this new graph is circuit-decomposable, and hence $K/\overline{B}$ itself has a decomposition into circuits and exactly one $q_{\overline{B}}$-ray.

Hence in both cases we obtain that B itself has a local circuit decomposition Δ_1. Let M be the union of the circuits of Δ_1 that are contained in B. Since $[B, \overline{B}]_K$ is finite, the vertices of B must all be of even degree in $K\backslash M$, and hence in $K\backslash(M \cup L)$ because $L \subseteq A-B$. Moreover, the vertices of $A-B$ are also of even degree in $K\backslash(M \cup L)$ because so are they in $K\backslash L$ and $M \subseteq B$. Thus the vertices of A ($= (A-B) \cup B$) are all of even degree in $K\backslash(M \cup L)$, and since $M \cup L$ is circuit-decomposable, A is therefore a locally circuit-decomposable region of K, and we are done.

Proof of the Claim By way of contradiction, assume that there is no finite cut of A separating O_A and D_A. By Menger's Theorem this implies that A contains an infinite set of edge-disjoint O_AD_A-paths. By Proposition 7.13(i), O_A is composed of an odd, and hence finite, number of odd-type classes of K. Therefore one of these classes, say U, is infinitely edge-connected (in A) to D_A. Since every odd-type class is also an ω-class it follows that for each vertex $u \in U$ there is an infinite family $\mathcal{P}_u$ of edge-disjoint uD_A-paths in K. $[A, \overline{A}]_K$ being finite, we may suppose without loss of generality that the paths in $\mathcal{P}_u$ belong to A. Denote by X_u the vertices in D_A that are endpoints of some path in $\mathcal{P}_u$.

Since $D_A \cap O_A = \emptyset$, and since any vertex which is infinitely edge-connected to an odd-type vertex is also of odd type, no vertex of D_A can belong to infinitely many different paths in $\mathcal{P}_u$, implying that the set X_u is infinite.

We now apply Proposition 2.11 to A. Since every ray of A is dominated in A it follows that in both situations covered by Proposition 2.11 we have a vertex $v \in V(A)$ and a family $\mathcal{L}_u$ of vX_u-paths of A which are vertex-disjoint except for their common endpoint v. It is easy to see that since each vertex in X_u dominates H in G, and since X_u is infinite, v also dominates H in G; thus $v \in D_A$. On the other hand, since clearly v and u are infinitely edge-connected in A, and $u \in O_A$, v also belongs to O_A, a contradiction. $\square$

9 Having enough eligible rays, a necessary and sufficient condition

Recall that having enough rays and having enough non-dominated rays are respectively a necessary and a sufficient condition for a graph to be circuit-decomposable. It turns out that in order to obtain a necessary and sufficient condition we have to define a new property of the rays of a graph which is a weakening of being non-dominated. It is tempting to consider removable rays, but unfortunately there exist graphs that have enough removable rays but are not circuit-decomposable, see the two examples on the second line of Figure 9. Since rays are never eulerian-type graphs, theorem 8.6 does not fully apply here. Note however that in those two examples of Figure 9, each removable ray is dominated by its own origin. Hence, from the circuit decomposition point of view, odd-type vertices of removable subgraphs are better to be as far from dominating vertices as possible.

Definition 9.1 A subgraph H of a graph G is *almost eulerian* if the set O_H of all the odd-type vertices of H and the set D_H of all the vertices that dominate H in G are separable by a finite cut of G. In other word, if there exists a finite cut $[A, \overline{A}]_G$ such that $D_H \in V(A)$ and $O_H \in V(\overline{A})$.

It is a consequence of Theorem 9.6 (see below) that having enough removable almost eulerian rays is a sufficient condition for circuit-decomposability. However this is not the property we are looking for, because there exist graphs that are circuit-decomposable but do not have enough removable almost eulerian rays (an example is shown in Figure 8 where the only removable ray is the one going to the left). Hence we have to consider a larger class of rays.

Definition 9.2 A ray R of G is said to be *eligible* if it is contained in a locally finite almost eulerian removable subgraph.

Observe that removable almost eulerian rays (and hence non-dominated rays) are eligible but the converse is not true in general. However, as is shown by the next result, there is only one other case to consider: the "removable 2-rays" (a 2-*ray* being a graph that is the union of two edge-disjoint rays having the same origin). Note also that in each example of the second line of Figure 9, there are enough rays that are contained in an almost eulerian removable subgraph. Hence the local finiteness condition in the definition of eligibility cannot be omitted if one want to characterize the circuit-decomposability.

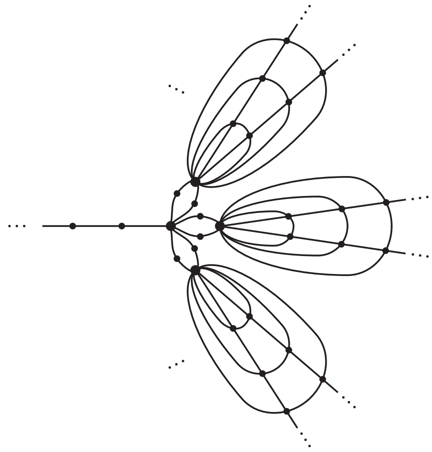

Figure 8:

Given any locally finite circuit-decomposable graph G and any x-ray R of G, the graph $G\backslash R$ is still locally finite and has exactly one vertex of odd degree (viz. x). This implies that $G\backslash R$ contains an x-ray R'. Since the subgraph $S := R \cup R'$ of G is a 2-ray, it follows from Theorem 2.24 that $G\backslash S$ is circuit-decomposable. Hence, in a locally finite circuit-decomposable graph any ray R is contained in a 2-ray that is a member of a decomposition of G into circuits and 2-rays. Since any such decomposition trivially induces a circuit decomposition, one might say that to construct a circuit decomposition of a locally finite circuit-decomposable graph, each ray is "usable". The next Proposition shows that in arbitrary circuit-decomposable graphs, each eligible ray is "usable".

Proposition 9.3 *Let R be an eligible u-ray of a circuit-decomposable graph G. Then there exists a u-ray R' contained in $G\backslash R$ such that $G\backslash(R \cup R')$ is still circuit-decomposable.*

Proof Let H be a locally finite almost eulerian removable subgraph of G that contains R. Let O_H be the set of all the odd-type vertices of H, D_H the set of all the vertices that dominate H in G, and $[A, \overline{A}]_G$ a finite cut of G such that $D_H \in V(A)$ and $O_H \in v(\overline{A})$.

For every vertex $x \in O_H$, define two new rays R_x^1 and R_x^2 such that R_x^1, R_x^2 and G pairwise intersect in x only and such that R_x^i and $R_{x'}^j$ are vertex-disjoint if $x \neq x'$. Let

$$\tilde{G} := G \cup \bigcup_{x \in O_H} (R_x^1 \cup R_x^2) \quad \text{and} \quad \tilde{H} := H \cup \bigcup_{x \in O_H} R_x^1.$$

Since H is locally finite, each vertex of O_H is of odd degree in H. This implies that $\tilde{H}$ is a locally finite eulerian graph, and hence an eulerian-type graph. Let us show that $\tilde{H}$ is also a removable subgraph of $\tilde{G}$. Since $V(R_x^i) \cap V(A) = \emptyset$ for any $x \in O_H$ and $i \in \{1, 2\}$, the finite cut $[A, \overline{A}]_G$ is also a cut of $\tilde{G}$. Hence $(\tilde{G}\backslash\tilde{H})/(\tilde{G} - A) = (G\backslash H)/(G - A)$, and it therefore follows from Lemma 7.9 that any vertex of A has same parity type in $G\backslash H$ and in $\tilde{G}\backslash\tilde{H}$. By construction, the set of vertices that dominate $\tilde{H}$ in $\tilde{G}$ is exactly D_H. This implies that each vertex of D_H is of even-type in $\tilde{G}\backslash\tilde{H}$ because $D_H \subseteq V(A)$ and because H is a removable subgraph of G. Thus $\tilde{H}$ is removable in $\tilde{G}$.

Since $\tilde{G}$ is clearly circuit-decomposable, by Theorem 8.6, there exists a circuit decomposition $\Delta_{\tilde{G}\backslash\tilde{H}}$ of $\tilde{G}\backslash\tilde{H}$. Let $\tilde{K}$ be $\Delta_{\tilde{G}\backslash\tilde{H}}$-shadow of $\bigcup_{x\in O_H} R_x^2$, and $K := \tilde{K} \cap G$. It is easy to see that $K \cup H$ is a locally finite eulerian subgraph of G. Hence $(K \cup H)\backslash R$ is locally finite and u is its only vertex of odd degree. This implies that there exists an u-ray R' contained in $(K \cup H)\backslash R$. Since $(K \cup H)\backslash(R \cup R')$ is still a locally finite eulerian graph, by Theorem 2.24, it has a circuit decomposition $\Delta_{(K\cup H)\backslash(R\cup R')}$ which, together with all the fragments of $\Delta_{\tilde{G}\backslash\tilde{H}}$ that are not contained in $\tilde{K}$, form a circuit decomposition of $G\backslash(R \cup R')$. □

Proposition 9.4 *Let A be an odd region of G. Then the following statements are equivalent.*

(i) *A contains an eligible ray;*

(ii) *$A \cup [A, \overline{A}]_G$ contains some almost eulerian removable ray or A contains some removable 2-ray;*

(iii) *A contains a non-dominated ray or a region B such that $G/\overline{B}$ has a decomposition into cycles and exactly one or exactly two $q_{\overline{B}}$-rays.*

Proof (ii) ⇒ (i) is evident because an almost eulerian ray and a 2-ray are always locally finite and almost eulerian subgraphs.

(iii) ⇒ (ii). Suppose A contains no non-dominated rays, and let B be a region that satisfies condition (iii). By the remark that follows the definition of a removable graph (Definition 8.3) and by the converse part of Lemma 8.4, we are done if $G/\overline{B}$ has a decomposition into cycles and exactly one $q_{\overline{B}}$-ray. Hence suppose that $G/\overline{B}$ has a decomposition Δ into cycles and exactly two $q_{\overline{B}}$-rays. B being a connected subgraph of $G/\overline{B}$, $q_{\overline{B}}$ is not a cut-vertex of $G/\overline{B}$. Hence, in B $(= G/\overline{B} - q_{\overline{B}})$, there exists a vertex v and two edge-disjoint v-rays R_1 and R_2 that have the tails of the two $q_{\overline{B}}$-rays of Δ. By Lemma 2.30, the set $\{R_1, R_2\}$ can be extended to a decomposition of $G/\overline{B}$ in which all the other fragments are cycles. Thus $G/\overline{B} - (R_1 \cup R_2)$ is cycle-decomposable. Since $q_{\overline{B}} \notin V(R_1 \cup R_2)$, again by the remark following Definition 8.3 and Lemma 8.4, we have that $R_1 \cup R_2$ is connected and removable in both $G/\overline{B}$ and G. Thus R_1 and R_2 are eligible in G.

(i) ⇒ (iii). Suppose that A contains dominated rays only. By Proposition 7.13 (i), A contains an odd number of odd-type class, say $O_1, O_2, \ldots, O_{2n+1}$, $(n \in \omega)$. Let R be any eligible ray of G contained in A, and u any of its dominating vertex. Since $[A, \overline{A}]_G$ is finite, $u \in V(A)$. Now let $H \supseteq R$ be any removable subgraph of G as given by Definition 9.2, and $[C, \overline{C}]_G$ be a finite cut of G such that $V(C)$ contains D_H (the set of vertices that dominate H in G) and $V(\overline{A})$ contains O_H (the set of vertices that are of odd degree in H). Observe that, by Lemma 2.2, $[A\cap C, \overline{A \cap C}]_G$ is finite. We now have to consider two cases.

Case 1. $u \notin \bigcup_{i=1}^{2n+1} O_i$. Then, by Lemma 2.3, there exists a region B of G that is contained in $A \cap C$, that contains u, and that is disjoint from $\bigcup_{i=1}^{2n+1} O_i$. By Theorem 7.20, B is a locally cycle-decomposable even region. Hence, $G/\overline{B}$ is cycle-decomposable. Moreover, since u dominates R, B contains a tail of R. Let Q be any such ray, and x be its origin.

Since no vertex of $H/\overline{B}$ except possibly $q_{\overline{B}}$ is of odd degree in $H/\overline{B}$, the latter is therefore locally finite and almost eulerian. Moreover, $H/\overline{B}$ is, by Lemma 8.4, removable in $G/\overline{B}$. Since Q is contained in $H/\overline{B}$, it is therefore an eligible ray of $G/\overline{B}$. Thus, it follows from Proposition 9.3 that there exists an x-ray Q' such that $G/\overline{B}$ has a decomposition $\Delta_{G/\overline{B}}$ into circuits and exactly two x-rays, the two x-rays being Q and Q'. Applying Theorem 2.24, let $\Delta_{Q \cup Q'}$ be a circuit decomposition of $Q \cup Q'$. Thus $\Delta := \Delta_{G/\overline{B}} \backslash \{Q, Q'\} \cup \Delta_{Q \cup Q'}$ is a circuit decomposition of $G/\overline{B}$.

If Δ contains at least a double-ray, then, by Lemma 3.8, $G/\overline{B}$ has a decomposition into circuits and exactly two $q_{\overline{B}}$-rays, as desired.

If Δ is a cycle decomposition, then Q and Q' are the only infinite fragments of $\Delta_{G/\overline{B}}$, and the existence of a decomposition of $G/\overline{B}$ into circuits and exactly two $q_{\overline{B}}$-rays is this time a consequence of Lemma 2.31 (replacing G, v, and Δ by $G/\overline{B}$, $q_{\overline{B}}$ and $\Delta_{G/\overline{B}}$, respectively).

Case 2. $u \in O_j$, for some j. Then, by Lemma 2.3, there exists a region D_{O_j} of G that is contained in $A \cap C$, that contains u (and hence O_j), and that is disjoint from $\bigcup_{i \neq j} O_i$. By Theorem 7.20, D_{O_j} is a peripheral region having exactly one odd-type class, viz O_j. Since $D_{O_j} \subseteq A$, without loss of generality we may suppose that $D_{O_j} = A$, and therefore that $O_H \subseteq V(\overline{A})$. By Corollary 7.21 and Lemma 8.4, to finish the proof, we only have to show that A is locally circuit-decomposable.

By Proposition 7.13 (i), there exists $v \in O_j \cap \mathrm{Bdry}_G(A)$ and hence some $v\overline{A}$-edge e. Let M be a connected graph (disjoint from $G/\overline{A}$) that has a cycle decomposition as well as a decomposition into cycles and exactly one ray S. Let Q be a ray disjoint from $G/\overline{A}$ and M, and form a new graph K from the union of $G/\overline{A}$, M and Q by identifying the vertices v and $q_{\overline{A}}$ of $G/\overline{A}$ with the origins of S and Q, respectively. K is circuit-decomposable because $M \cup e \cup Q$ obviously is circuit-decomposable, and $(G/\overline{A})\backslash e$ is cycle-decomposable by Proposition 7.19. We leave it to the reader to show that O_j is still the only odd-type class of K, and that $(H/\overline{A}) \cup M$ and $(H/\overline{A}) \cup M \cup Q$ are both removable in K (here one uses that v already dominates $H/\overline{A}$ in $G/\overline{A}$). Since $O_H \subseteq \overline{A}$, one of $H/\overline{A}$ and $(H/\overline{A}) \cup Q$ is eulerian, we therefore have by Theorem 8.6 that one of

$$K \backslash ((H/\overline{A}) \cup M) \text{ and } K \backslash ((H/\overline{A}) \cup M \cup Q)$$

is still circuit-decomposable. In either case this implies that $(G/\overline{A})\backslash(H/\overline{A})$ has a decomposition into circuits and $q_{\overline{A}}$-rays and therefore, since $H/\overline{A}$ is locally finite and has no vertex of odd degree except possibly $q_{\overline{A}}$, that $G/\overline{A}$ itself admits a decomposition into circuits and $q_{\overline{A}}$-rays. Thus A is locally circuit-decomposable, and we are done. □

The following result is our main theorem.

Theorem 9.5 *A graph is circuit-decomposable if and only if it has enough eligible rays.*

Proof The necessity is a direct consequence of Proposition 7.18, Corollary 7.21, and Proposition 9.4. For the sufficiency, by Proposition 7.18, we have to show that each obstructive peripheral region that contains exactly one odd-type class is locally circuit-decomposable. Let G be a graph and A any such region, O the only odd-type class in A, and K the graph obtained from $G/\overline{A}$ by attaching a new ray R_0 to the vertex $q_{\overline{A}}$. To finish the proof, let us show that K is circuit-decomposable. Lemma 7.9, applied to A as induced subgraph of G, and again to A as induced subgraph of K shows that the type of the vertices of A is the same in G, $G/\overline{A}$, and K. Thus, O is also the unique odd-type class of K. Moreover, since G has enough eligible rays and since R_0 is non-dominated (and hence eligible) in K, Lemma 8.4 implies that K has enough eligible rays.

Case 1. There exists a region B of G contained in A such that $G/\overline{B}$ has a decomposition into cycles and exactly one $q_{\overline{B}}$-ray (in particular, B is locally circuit-decomposable). Then B must be an odd region and therefore, by Theorem 7.20, a peripheral region of G (and hence also of K). Moreover, since K has exactly one odd-type class, it follows from Proposition 7.13 (i) that K contains no peripheral region that is disjoint from B. Hence $\{B\}$ is a maximal family of vertex-disjoint obstructive peripheral region, implying, by Theorem 7.5 ((iv) $\Rightarrow$ (i)), K is circuit-decomposable.

Case 2. There is no region having the properties of Case 1. Then by Proposition 9.4 ((i) $\Rightarrow$ (iii)) there is a non-empty family $(B_i)_{i\in I}$ of vertex-disjoint regions of G that are contained in A such that each $G/\overline{B_i}$ has a decomposition Δ_i into cycles and exactly two $q_{\overline{B}_i}$-rays. Suppose that $(B_i)_{i\in I}$ is maximal with respect to these properties, and note that each B_i must be an even region and hence, by Theorem 7.20, is disjoint from O. We claim that in G (and hence also in K), O cannot be separated from $\bigcup_{i\in I} B_i$ by a finite cut. Indeed, if there exists a region C of G that contains O and is disjoint from $\bigcup_{i\in I} B_i$, then the component of $C - \overline{A}$ that contains O is, by Theorem 7.20, a peripheral region of G. Since G has enough eligible rays, Proposition 9.4 ((i) $\Rightarrow$ (iii)) implies that either there exists a region having the properties of Case 1, or the family $(B_i)_{i\in I}$ is not maximal; in either case we reach a contradiction.

For each $i \in I$ let L_i be the Δ_i-shadow of some finite connected subgraph of $G/\overline{B_i}$ that contains all the edges incident to the vertex $q_{\overline{B}_i}$ and in which $q_{\overline{B}_i}$ is not a cut-vertex. Such an L_i exists because, B_i being connected, $q_{\overline{B}_i}$ is not a cut-vertex of $G/\overline{B_i}$ in the first place. Then let K_i be the graph obtained from $(G/\overline{B_i})\backslash L_i$ by deleting isolated vertices. Observe that these graphs have the following properties:

- $q_{\overline{B}_i}$ does not belong to K_i (or equivalently $K_i \subseteq B_i$);
- L_i contains at least two edge-disjoint rays, namely the two $q_{\overline{B}_i}$-rays of Δ_i;
- K_i is cycle-decomposable;
- $L_i - q_{\overline{B}_i}$ is connected and locally finite.

Now consider $L := K\backslash(\bigcup_{i\in I} K_i)$. Note that

$$E(L \cap B_i) \cup [q_{\overline{B}_i}, B_i]_{G/\overline{B}_i} = E(L_i). \tag{3}$$

To finish the proof, it suffices to show that L has a circuit decomposition. Suppose the contrary. By Proposition 7.18, there is an obstructive peripheral region D of L that contains exactly one odd-type class of the graph L, say O_D, and which is not locally circuit-decomposable. By (3), the vertices of each B_i have the same degree in L as in L_i, and since L_i is the shadow of a finite subgraph of $G/\overline{B}_i$, each $x \in V(B_i)$ has even degree in L_i, and hence in L. Therefore none of the vertices of B_i, $i \in I$, belongs to O_D.

Note that $[L_i - q_{\overline{B}_i}, \overline{L_i - q_{\overline{B}_i}}\,]_L$ is finite because it is equal to $[B_i, \overline{B_i}]_K$. Since moreover, $L_i - q_{\overline{B}_i}$ is locally finite, each ray contained in L_i is non-dominated in L. Hence each ray of each Δ_i ($i \in I$) induces a non-dominated ray in L, and none of these has a tail in D. Therefore by (3) and the connectedness of $L_i - q_{\overline{B}_i}$, it follows that if a vertex of B_i belongs to D, then at least one edge of B_i belongs to $[D, \overline{D}]_L$, for any $i \in I$. Since $[D, \overline{D}]_L$ is finite, so is $V(D) \cap (\bigcup_{i\in I} V(B_i))$. Moreover, D being an obstructive peripheral region of L, at most a finite initial segment of the ray $R_0^+ := [q_{\overline{A}}, G/\overline{A} - q_{\overline{A}}]_K \cup R_0$ belongs to D. Thus,

$$V(D) \cap \Big(V(R_0^+) \cup \bigcup_{i\in I} V(B_i)\Big) \text{ is finite.}$$

By Lemma 2.3, $V(R_0^+) \cup \bigcup_{i\in I} V(B_i)$ is therefore separable from O_D by a finite cut in L. Let $N \subseteq D$ be any region of L that contains the vertices of O_D but no vertex of $V(R_0^+) \cup \bigcup_{i\in I} B_i$, and observe that N is also contained in A ($= K - V(R_0^+)$) and that

$$[N, K - N]_K = [N, L - N]_L. \tag{4}$$

Moreover, N cannot contain O because, as proved earlier, O is not separable in K from $\bigcup_{i\in I} V(B_i)$ by a finite cut. Since A is peripheral in K and does not contain O, by Theorem 7.20, N is therefore an even region of K. On the other hand, again by Theorem 7.20, N is an odd region of L because it is contained in D (which is peripheral in L) and contains O_D, a contradiction to (4). □

Theorem 9.5 can be stated in this apparently stronger form.

Theorem 9.6 *A graph G is circuit-decomposable if and only if every peripheral region contains an eligible ray.*

Proof Theorem 9.5, Proposition 7.4 and the fact that a non-dominated ray is always eligible. □

In the remainder of this section we will show that "most" of the ends of a graph contain eligible rays. This will imply that eulerian graphs that are not circuit-decomposable must have a very particular structure. The next two propositions will make this statement more explicit.

Proposition 9.7 *Every end of G that is of ϵ-multiplicity > 2 contains some eligible ray.*

Proposition 9.8 *Every end of G that is of ϵ-multiplicity > 1 and is ϵ-dominated by an odd-type vertex of the graph contains some eligible ray.*

We first prove Proposition 9.8 as it will be used in the proof of Proposition 9.7.

Proof of Proposition 9.8 Let τ be such an end, O the odd-type class whose members ϵ-dominate τ, and A be a peripheral region that contains O and no other odd-type class of G. By Remark 7.17 such an A exists. Clearly, each ray of τ has a tail in A. By Proposition 7.13 (i), there exists $u \in O \cap \mathrm{Bdry}_G(A)$; let $e \in [u, \overline{A}]_G$. Note that by Proposition 7.19 there exists a cycle decomposition Δ of the graph $(G\backslash e)/\overline{A}$. Since the ϵ-multiplicity of τ is greater than 1, then by Theorem 3.3, $(G\backslash e)/\overline{A}$ has a decomposition Δ' into cycles and exactly one u-ray, say R. Observe that $R \cup e$ is either a ray or the edge-disjoint union of a tail of R and a cycle. Without loss of generality we may suppose that $R \cup e$ is a ray because in the other possible case, we can take that specific tail of R instead of R and then add the extra cycle so obtained to Δ'.

Now, since $\Delta'\backslash\{R\}$ is a cycle decomposition of $(G/\overline{A})\backslash(R \cup e)$, it follows from Corollary 7.8 that $(G/\overline{A})\backslash(R \cup e)$ has no vertex of odd-type. Hence $R \cup e$ is removable in $G/\overline{A}$ which, by Lemma 8.4, implies that the ray S of G induced by the edges of $R \cup e$ is removable in G. Since moreover the origin of S is in $\overline{A}$, S is therefore an eligible ray contained in τ. $\square$

Proof of Proposition 9.7 Without loss of generality assume G to be connected. By way of contradiction, suppose that there exists an end τ of ϵ-multiplicity > 2 that contains no eligible ray. Since τ must be dominated, the set X that consists of all the vertices of G that ϵ-dominate τ in G is non-empty and hence is an ω-class. Moreover, by Proposition 9.8, X is an even-type class. Now there are two cases to consider.

Case 1. X is contained in some peripheral region A of G. Since by Proposition 7.13 (i), A contains only finitely many odd-type classes, it follows from Lemma 2.3 that there exists a region $B \subseteq A$ which contains X but no odd-type class of G. Note that each ray of τ has a tail in B and that by Proposition 7.12, B is locally cycle-decomposable. Hence $G/\overline{B}$ is cycle-decomposable which implies by Theorem 3.3 that given any $x \in X$, $G/\overline{B}$ has a decomposition into cycles and exactly two x-rays $R_1, R_2 \in \tau$. By Lemma 2.30 we may suppose that neither R_1 nor R_2 contains the vertex $q_{\overline{B}}$. This implies that $R_1 \cup R_2$ is removable in $G/\overline{B}$ and hence, by Lemma 8.4, also removable in G. Thus both R_1 and R_2 are eligible rays.

Case 2. X is not contained in any peripheral region of G. Let u be any vertex that dominates τ in G (hence $u \in X$), and consider $H := R_1 \cup R_2 \cup R_3 \cup R_{12} \cup R_{13} \cup \bigcup_{i\in\omega} P_i$, where R_1, R_2, R_3 are three edge-disjoint u-rays of τ, and R_{12} and R_{13} are u-rays of τ such that R_{1j} meets both R_1 and R_j infinitely often, $j = 2, 3$, and $(P_i)_{i\in\omega}$ is an infinite family of uR_1-paths having only u in common. Let

$(A_i)_{i\in I}$ be a maximal family of (vertex-)disjoint odd regions of G that are (vertex-)disjoint from H, and put $A := \bigcup_{i\in I} A_i$.

Construct a new (multi-)graph $\widetilde{G}$ as follows: add a new vertex q to G and ω new qy-edges for each odd type vertex y of G that is not in A, and then identify all the vertices of A with q (deleting loops). The vertex of $\widetilde{G}$ resulting from this identification will be denoted by $\widetilde{q}$. Note that in $\widetilde{G}$, the odd type vertices of G have either been identified with $\widetilde{q}$ or are joined to $\widetilde{q}$ by infinitely many edges.

We claim that every odd region of $\widetilde{G}$ contains $\widetilde{q}$. By way of contradiction let B be an odd region of $\widetilde{G}$ such that $\widetilde{q} \notin V(B)$. Being contained in $G - A$ $(= \widetilde{G} - \widetilde{q})$, B is also an odd region of G. Since B cannot contain any odd type vertex of G, it follows from Proposition 7.13 (i) that B contains no peripheral region of $\widetilde{G}$. By Proposition 7.2, B therefore contains an ϵ-stratifying sequence $(B_n)_{n\in\omega}$ of odd regions of $\widetilde{G}$. Since each ray contained in H is dominated (and hence ϵ-dominated) in G, by Corollary 2.18 there exists an $n_0 \in \omega$ such that $V(H) \cap V(B_{n_0}) = \emptyset$. This is a contradiction to the maximality of the family $(A_i)_{i\in I}$ because $B_{n_0} \subseteq B$, and B is disjoint from A $(= \bigcup_{i\in I} A_i)$. Thus every odd region of $\widetilde{G}$ contains $\widetilde{q}$, as claimed.

It follows from this claim that $\widetilde{G}$ has no odd cut, and hence is cycle-decomposable by Nash-Williams's Theorem. Since $H \subseteq \widetilde{G}$, u ϵ-dominates $\widetilde{\tau}$ in $\widetilde{G}$, and by Theorem 3.3 (applied to $\widetilde{G}$ and to the end $\widetilde{\tau}$ of $\widetilde{G}$ that contains the ray R_1), $\widetilde{G}$ has a decomposition Δ into cycles and exactly two u-rays belonging to $\widetilde{\tau}$ (R and R', say). Since u ϵ-dominates $\widetilde{\tau}$ also in $\widetilde{G} - \widetilde{q}$, there exist two edge-disjoint u-rays of $\widetilde{G}$ that have tails in $\{R, R'\}$ and do not contain the vertex $\widetilde{q}$. Hence, by Lemma 2.30, we may suppose that Δ has been so chosen that neither R nor R' contains $\widetilde{q}$, or in other words that R and R' belong to G. A being disjoint from H, R and R', it is easy to see that R and R' are both end-equivalent to R_1 in G. Hence $R, R' \in \tau$.

Put $Q := R \cup R'$. Clearly Q is a locally finite eulerian graph of G. To complete the proof, let us show that Q is removable in G. By way of contradiction, suppose that there is a vertex x that dominates Q in G and is of odd type in $G\backslash Q$. Note that x belongs to X, and hence is of even type in G; also, being an ϵ-dominating vertex of each ray in H, x does not belong to any A_i. Let O_x be the ω-class of x in $G\backslash Q$, and applying Remark 7.17, choose a peripheral region C of $G\backslash Q$ such that the only odd type class of $G\backslash Q$ contained in C is O_x.

Since every vertex that ϵ-dominates Q in G belongs to X and hence is of even type in G, it follows from Proposition 7.10 that each vertex that is of odd type in G is also of odd type in $G\backslash Q$. Since $x \in X$ (which is an even type class of G), no vertex that is of odd type in G may belong O_x. Thus C contains no vertex that is of odd type in G.

The A_i's are odd regions of $G\backslash Q$ because they are odd regions of G and $V(A_i) \cap V(Q) = \emptyset$. Moreover, if $V(A_i) \cap V(C) \neq \emptyset$ then, A_i being connected,

$$\text{either } \ A_i \subseteq C \ \text{ or } \ E(A_i) \cap [C, \overline{C}]_{G\backslash Q} \neq \emptyset.$$

The first possibility implies that

$$[A_i, (G\backslash Q) - A_i]_{G\backslash Q} \cap [C, \overline{C}]_{G\backslash Q} \neq \emptyset$$

because C is peripheral in $G\backslash Q$. On the other hand, the A_i's being disjoint, each edge $e \in [C, \overline{C}]_{G\backslash Q}$ can belong to $E(A_i)$ for at most one $i \in I$, or to $[A_i, (G\backslash Q) - A_i]_{G\backslash Q}$ for at most two $i \in I$. Thus the set J of all the indices $i \in I$ such that $V(A_i) \cap V(C) \neq \emptyset$ is finite.

If $J = \emptyset$, put $C' := C$, otherwise let $K := \bigcup_{j \in J} A_j$ and C' be the component of $C - K$ that contains x (recall that x does not belong to any A_i). Since $[K, \overline{K}]_{G\backslash Q}$ is clearly finite, so is $[C - K, \overline{C - K}]_{G\backslash Q}$ by Lemma 2.2. Hence C' is a region of $G\backslash Q$. Moreover, since O_x is the only odd type class of $G\backslash Q$ in C, it follows from Theorem 7.20 (with G, O, B, A replaced by $G\backslash Q$, O_x, C', C, respectively) that C' is an odd region of $G\backslash Q$.

The region C' is disjoint from A ($= \bigcup_{i \in I} A_i$). Therefore in the passage from G to $\widetilde{G}$, no vertex of C is identified with q, so that C' is contained in $\widetilde{G} - \widetilde{q}$. Moreover, every edge of G incident with a vertex of C' is also an edge of $\widetilde{G}$ (possibly with one of its endpoints identified with q). On the other hand, being a subset of C, C' does not contain any vertex that is of odd type in G. This means that in the passage from G to $\widetilde{G}$, no edges are added at any vertex of C'. Hence the cuts induced by C' in $G\backslash Q$ and $\widetilde{G}\backslash Q$ coincide, which means that C' is an odd region of $\widetilde{G}\backslash Q$, a contradiction to Nash-Williams's Theorem because $Q = R \cup R'$ and $\Delta - \{R, R'\}$ is a cycle decomposition of $\widetilde{G}\backslash(R \cup R')$. □

From Proposition 9.7 and Remark 2.4, we obtain the following corollary.

Corollary 9.9 *An end which admits infinitely many dominating vertices contains an eligible ray.*

Moreover, it follows from Propositions 9.7 and 9.8 that in a graph which is not circuit-decomposable there must exist a peripheral region in which ends are very "thin". We generalize this idea in Proposition 9.11, which is based on the following definition.

Definition 9.10 An end τ of a graph G is said to be *threadlike* if it has ϵ-multiplicity 1 and at least two dominating vertices.

By Remark 2.4, a threadlike end has at most finitely many dominating vertices.

The next proposition says that in a region that has no eligible ray, "almost" all ends are threadlike and the exceptions are "almost threadlike" in the sense that they satisfy only one of the two defining conditions of threadlike ends. We shall use the following notation: for a vertex $x \in V(G)$ denote by $\mathcal{D}_x$ the set of all ends of G that are ϵ-dominated by x in G.

Proposition 9.11 *Let A be a peripheral region of G that contains no eligible ray. Then all rays in A are dominated, and given any vertex $x \in A$ of infinite degree, either*

(i) *x is of odd type and all ends in $\mathcal{D}_x$ are threadlike*

or

(ii) *x is of even type and all but at most one end in $\mathcal{D}_x$ are threadlike, and the exceptional end τ (if it exists) has one of the following properties:*

(1) *τ has ϵ-multiplicity 1 and exactly one dominating vertex;*

(2) *τ has ϵ-multiplicity 2 and at least two dominating vertices.*

Proof It is a direct consequence of the definition of eligibility that all rays in A are dominated.

Case (i): x is of odd type. Let $\tau \in \mathcal{D}_x$. Then by Proposition 9.8, τ is of ϵ-multiplicity 1. Moreover, by exactly the same argument as in the proof of Proposition 9.8 except that Theorem 3.3 is replaced by Proposition 3.6, it follows easily that τ cannot be dominated by exactly one vertex. Thus τ is threadlike.

Case (ii): x is of even type. Suppose there are two distinct ends τ, τ' in $\mathcal{D}_x$ that are not threadlike.

As in the proof of Proposition 9.7 (case 1), choose a locally cycle-decomposable region B of G that is contained in A and that contains x but no odd-type class of G. Thus every end in $\mathcal{D}_x$ has rays in B. Let u be any vertex that dominates τ in G. Clearly u is in B. Observe that u is infinitely edge-connected to x and hence ϵ-dominates τ' in G (and therefore in $G/\overline{B}$), and that for any ray $R \in \tau'$, u still dominates τ in $G\backslash R$ and therefore also in $(G/\overline{B})\backslash(R/\overline{B})$). Now apply either Theorem 3.3 or Proposition 3.6 in order to construct a decomposition of $G/\overline{B}$ into cycles and exactly one u-ray $R' \in \tau'$. Observe that $\widetilde{G} := (G/\overline{B})\backslash R'$ is cycle-decomposable and that, τ and τ' being different ends of G, the set $\widetilde{\tau}$ of all rays of τ that are contained in $\widetilde{G}$ is an end in $\widetilde{G}$. It is easy to see that $\widetilde{\tau}$ is not threadlike and still dominated by u in $\widetilde{G}$.

Again, apply either Theorem 3.3 or Proposition 3.6 to construct a decomposition of $\widetilde{G}$ into cycles and exactly one u-ray $\widetilde{R} \in \widetilde{\tau}$. Clearly $D := R' \cup \widetilde{R}$ is a removable 2-ray in $G/\overline{B}$ because D is locally finite and eulerian and $(G/\overline{B})\backslash D$ is cycle-decomposable. By Lemma 8.4, D is also removable in G. R' and $\widetilde{R}$ are therefore eligible rays of G contained in A, a contradiction to the hypothesis of the proposition.

To finish the proof consider a non-threadlike end τ that has rays in A. By Proposition 9.7, τ has ϵ-multiplicity ≤ 2. By an argument that follows verbatim the proof of Proposition 9.7 (case 1) except that we use Proposition 3.6 instead of Theorem 3.3, one may once again easily show that every end of ϵ-multiplicity $= 2$ that has rays in A must be dominated by more than one vertex of G. This implies that one of properties (1) and (2) holds for the end τ. □

Proposition 9.12 *A graph having at most one vertex of infinite degree is circuit-decomposable if and only if it has enough rays.*

Proof The necessity has already been proved in Proposition 4.2. For the sufficiency, suppose G is a counterexample. Having enough rays, G cannot contain vertices of odd degree, but by Corollary 7.15, it contains a vertex of odd-type. Hence G contains exactly one vertex of infinite degree, and that vertex is of odd-type in G. Denote that vertex by x. Apply Theorem 9.6 to choose a peripheral region A that contains no eligible rays. By Proposition 7.13 (i), x belongs to A. Since x is the only vertex of infinite degree, no end of A is threadlike, but x must dominate (and therefore ϵ-dominate) each ray of A. Thus, by Proposition 9.11, the odd region A must be rayless, contradicting the hypothesis. □

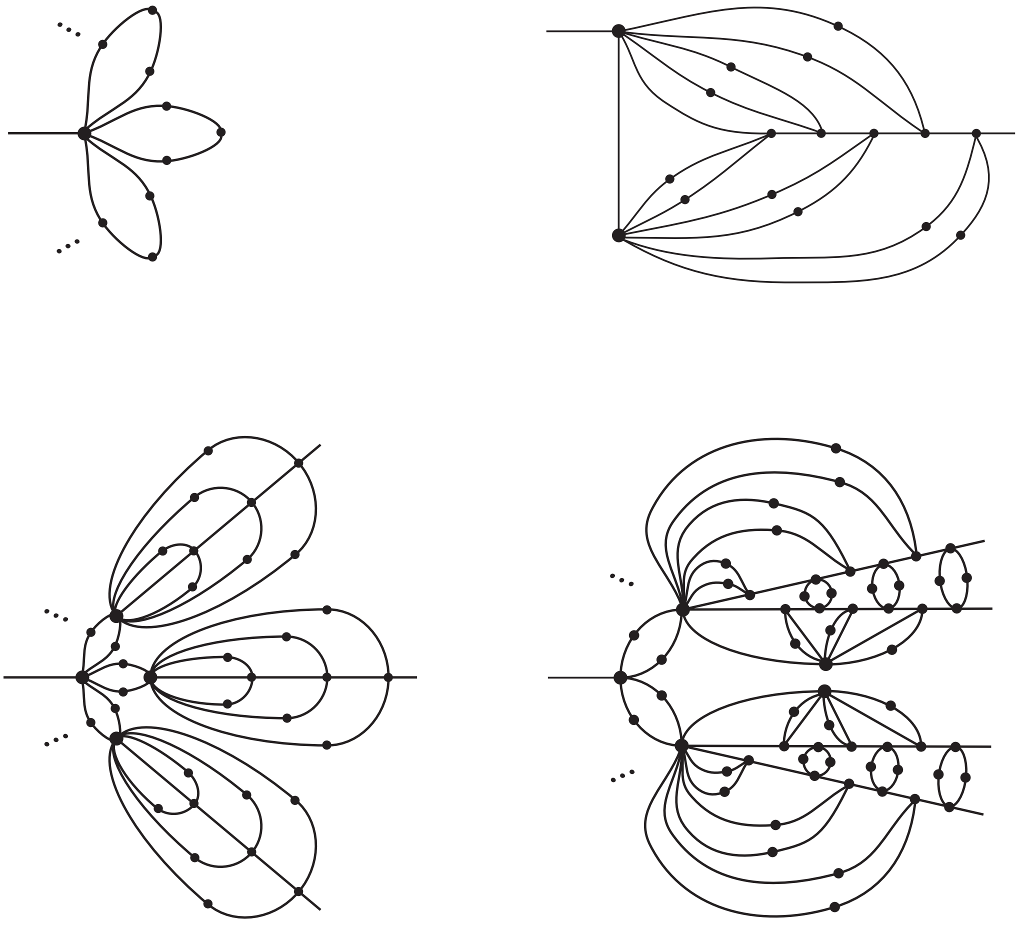

Figure 9: Peripheral regions that are not locally circuit-decomposable

It follows from Theorem 9.6, Proposition 7.13 (i) and Proposition 9.11 that a graph which is not circuit-decomposable, must contain an odd-type vertex that ϵ-dominates threadlike ends only. Moreover, those results together with Proposition 7.13 (ii) say that if a graph is not circuit-decomposable, then it must contain a peripheral region which has exactly one odd-type class, which is in some sense obtained by identifying vertices of some of the four types of odd regions shown in Figure 9. Each of these four regions is obviously peripheral, contains exactly one odd-type class and is not locally circuit-decomposable.

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