Decomposition of infinite eulerian graphs with a small number of vertices of infinite degree

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Abstract

We consider the question whether an infinite eulerian graph has a decomposition into circuits and rays if the graph has only finitely many, say $n$, vertices of infinite degree, and only finitely many finite components after the removal of the vertices of infinite degree. It is known that the answer is affirmative for $n < 2$ and negative for $n \geq 4$. We settle the remaining case $n = 3$, showing that a decomposition into circuits and rays also exists in this case.

1. Preliminaries

In this paper we shall deal with a special case of the problem of decomposing infinite eulerian graphs (i.e., graphs whose vertices are of even or infinite degree) into edge disjoint (finite or infinite) circuits and rays (one-way infinite paths). In general an eulerian graph does not admit such a decomposition. Sabidussi [2] raised the question whether a circuit-ray decomposition exists under the additional assumptions that the graph

1. has only finitely many, say $n$, vertices of infinite degree and
2. has only finitely many finite components after the removal of the vertices of infinite degree.

It is easily seen that for $n \leq 2$ this is indeed the case (see [2]). On the other hand, Thomassen [3] gave an example (Fig. 1) showing that for $n = 4$ a circuit-ray decomposition need not exist. Thomassen's counterexample is easily generalized to arbitrary $n \geq 4$. Thus there remains the case $n = 3$. The purpose of this note is to prove that in this case the answer is affirmative.

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Definitions For convenience, all graphs considered in this paper are without multiple edges or loops. Given an arbitrary graph $G$ we denote by $I_G$ the set of all vertices of infinite degree of $G$, and by $J_G$ the set of all vertices whose degree is either odd or infinite. $G$ is eulerian if all vertices are of even or infinite degree (we do not require connectedness). A circuit is a nonempty, connected, 2-regular graph, a cycle is a finite circuit. A ray is a one-way infinite path; its unique vertex of degree 1 is its origin. A ray whose origin is $x$ will be called an $x$-ray.

A decomposition of a graph $G$ is a set of pairwise edge-disjoint subgraphs of $G$ whose union is $G$. A CR-decomposition is a decomposition consisting of circuits and rays.

Given a graph $G$ and a subgraph $H$ of $G$ we denote by $G \setminus H$ the subgraph of $G$ consisting of the edges of $G$ which are not in $H$, and their incident vertices (i.e., the edge-induced subgraph). Note that by definition $G \setminus H$ never has an isolated vertex. For $A \subseteq V(G)$ we denote by $[A, \bar{A}]$ the set of all edges of $G$ having one vertex in $A$ and the other in $\bar{A} = V(G) \setminus A$. A set of the form $[A, \bar{A}]$ will be called a cut of $G$.

We will use the following two classical theorems.

**König's Theorem** (König [1]). Let $G$ be an infinite, locally finite connected graph. Then for any $x \in V(G)$, there exists an $x$-ray in $G$.

**Veblen's Theorem** (König [1]). Let $G$ be a locally finite eulerian graph. Then $G$ has a circuit decomposition.

2. Results

**Lemma 2.1** (folklore). If $F$ is an infinite rayless forest without isolated vertices then $J_G$ is infinite.
Proof. \( F \) has infinitely many pendant vertices. \( \square \)

**Lemma 2.2.** If \( G \) is a graph with at most one vertex of odd or infinite degree, then \( G \) has a CR-decomposition.

**Proof.** If there is no such vertex we are in the case of Veblen's theorem. Suppose, then, that \( G \) has a unique vertex \( x_0 \) whose degree is odd or infinite. Let \( \mathcal{D} \) be a maximal set of pairwise edge-disjoint circuits and \( x_0 \)-rays. We claim that \( \mathcal{D} \) is a decomposition of \( G \). Consider \( D = \bigcup \mathcal{D} \). By the maximality of \( \mathcal{D} \) and the fact that all vertices of \( G \) except \( x_0 \) are of even degree, it follows that \( G \setminus D \) is rayless, acyclic, without isolated vertices, and has at most one vertex of odd or infinite degree, namely \( x_0 \). Since any nonempty rayless forest without isolated vertices has at least two pendant vertices, we therefore obtain that \( G \setminus D = \emptyset \). Thus \( \mathcal{D} \) is a CR-decomposition of \( G \). \( \square \)

**Lemma 2.3.** Let \( G \) be a graph such that \( I_G \) is finite and \( G - I_G \) has only finitely many finite components. Then any \( x \in I_G \) is the origin of a ray in \( G \).

**Proof.** Let \( x \in I_G \). Since \( x \) has infinite degree and the number of finite components of \( G - I_G \) is finite, \( x \) has a neighbor \( y \) in some infinite component \( H \) of \( G - I_G \). \( H \) being locally finite, \( y \) is the origin of some ray \( R \subset H \) (by König's theorem). Thus the edge \((x, y)\) together with \( R \) form an \( x \)-ray in \( G \). \( \square \)

The following lemma can also be proved without any restrictions on the cardinality of \( I_G \). We consider here only the case where \( I_G \) is finite or countable as this is all we need in the sequel.

**Lemma 2.4.** Let \( G \) be an eulerian graph having at most countably many vertices of infinite degree. If \( G \) has no CR-decomposition, then there is a finite cut \([A, \overline{A}]\) of \( G \) which separates some vertices of infinite degree, i.e., \( A \cap I_G \neq \emptyset \) and \( \overline{A} \cap I_G \neq \emptyset \).

**Proof.** Suppose by way of contradiction that for any finite cut \([A, \overline{A}]\) of \( G \) either \( I_G \subset A \) or \( I_G \subset \overline{A} \). This implies that given any finite subgraph \( F \) of \( G \), \( I_G \) is contained in some component of \( G \setminus F \).

Note that by Lemma 2.2 \(|I_G| \geq 2\). Since \( I_G \) is finite or countable, we can form a countable sequence \( p_0, p_1, \ldots \) of pairs of distinct vertices of \( I_G \), say \( p_i = \{x_i, y_i\} \), in which every pair of distinct vertices of \( I_G \) occurs infinitely often. Using the pairs \( p_i \), construct an infinite sequence \( P_0, P_1, \ldots \) of pairwise edge-disjoint paths as follows. Let \( P_0 \) be any \( x_0 \)-path in \( G \), and assuming \( P_0, \ldots, P_n \) already constructed, let \( P_{n+1} \) be an \( x_{n+1} \)-path in \( G \setminus (P_0 \cup \cdots \cup P_n) \).

Extend the set \( \{P_0, P_1, \ldots\} \) to a maximal set \( \mathcal{D} \) of pairwise edge-disjoint paths in \( G \) having both endpoints in \( I_G \). It follows from the choice of the pairs \( p_i \) that given any two distinct vertices \( x, y \in I_G \) there are infinitely many \( xy \)-paths in \( \mathcal{D} \).
Let $D = \bigcup \mathcal{D}$ and consider $G \setminus D$. As the vertices of odd or infinite degree of $G \setminus D$ are among the vertices of $I_G$, the maximality of $\mathcal{D}$ implies that each component of $G \setminus D$ has at most one vertex of odd or infinite degree. Hence by Lemma 2.2 each component of $G \setminus D$ has a CR-decomposition and therefore so does $G \setminus D$.

To complete the proof we show that $D$ has a decomposition into cycles. Given $x, y \in I_G, x \neq y$, let $D_{xy}$ be the union of all $xy$-paths in $\mathcal{D}$. As already mentioned there are infinitely many such paths. They can be paired to form finite eulerian graphs all of whose vertices are of degree 2 or 4. These can be decomposed into cycles and hence give rise to a cycle decomposition of $D_{xy}$. Moreover, the graphs $D_{xy}, x, y \in I_G, x \neq y,$ form a decomposition of $D$. Hence combining the cycle decompositions of the $D_{xy}$’s we obtain a cycle decomposition of $D$. $\square$. 

**Theorem 2.5.** Let $G$ be a graph such that $|I_G| = 3$ and $G - I_G$ has only finitely many finite components. Then $G$ has a CR-decomposition.

**Proof.** Suppose $G$ has no such decomposition. Let $I_G = \{x_1, x_2, x_3\}$. By Lemma 2.4 there is a finite cut $[A, \bar{A}]$ such that w.l.o.g. $x_1 \in A$ and $x_2, x_3 \in \bar{A}$.

Denote by $G_1, G_2$ the induced subgraphs of $G$ on $A$ and $\bar{A}$, respectively, and abbreviate $I_G$ by $I_i, i = 1, 2$. Clearly $I_1 = \{x_1\}$ and $I_2 = \{x_2, x_3\}$. Moreover, since $[A, \bar{A}]$ is finite, both $G_1 - I_1$ and $G_2 - I_2$ have only a finite number of finite components. Therefore by Lemma 2.3, $G_i$ contains an $x_i$-ray $R_i (i = 1, 2)$, and obviously $R_1$ and $R_2$ are disjoint. Let $H = G \setminus (R_1 \cup R_2)$.

Consider a maximal set $\mathcal{D}$ of pairwise edge-disjoint circuits and $x_i$-rays ($i = 1, 2, 3$) in $H$, and let $D = \bigcup \mathcal{D}$. We will show that $\mathcal{D}$ can be extended to a CR-decomposition of $G$, i.e., that $G \setminus D$ has a CR-decomposition.

Observe that $H \setminus D$ is acyclic and rayless and that $J_{H \setminus D} = \{x_1, x_2, x_3\}$. Hence by Lemma 2.1, $H \setminus D$ is finite and the only vertices which may have odd degree are $x_1, x_2, x_3$. Moreover, since any finite graph has an even number of vertices of odd degree we conclude that $G \setminus D$ has at most two vertices of odd degree, because otherwise $H \setminus D$ would have only one, viz. $x_3$. There are now two cases.

**Case 1.** $G \setminus D$ has at most one vertex of odd degree. Then by Lemma 2.2, $G \setminus D$ has a CR-decomposition.

**Case 2.** $G \setminus D$ has exactly two vertices of odd degree. This means that at least one of $x_1$ and $x_2$ has odd degree in $G \setminus D$, say $x_1$. Since $R_1$ is an $x_1$-ray, $(G \setminus D) \setminus R_1$ has a CR-decomposition (Lemma 2.2), and hence so does $G \setminus D$.

Thus in either case we reach a contradiction. $\square$

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References