Almost Sure Bisimulation in Labelled Markov Processes

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Abstract

In this paper we propose a notion of bisimulation for labelled Markov processes parameterised by negligible sets (LMPns). The point is to allow us to say things like two LMPs are "almost surely" bisimilar when they are bisimilar everywhere except on a negligible set. Usually negligible sets are sets of measure 0, but we work with abstract ideals of negligible sets and so do not introduce an ad-hoc measure.

The construction is given in two steps. First a refined version of the category of measurable spaces is set up, where objects incorporate ideals of negligible subsets, and arrows are identified when they induce the same homomorphisms from their target to their source σ-algebras up to negligible sets. Epis are characterised as arrows reflecting negligible subsets. Second, LMPns are obtained as coalgebras of a refined version of Giry's probabilistic monad. This gives us the machinery to remove certain counterintuitive examples where systems were bisimilar except for a negligible set. Our new notion of bisimilarity is then defined using cospans of epis in the associated category of coalgebras, and is found to coincide with a suitable logical equivalence given by the LMP modal logic. This notion of bisimulation is given in full generality - not restricted to analytic spaces. The original theory is recovered by taking the empty set to be the only negligible set.

Some of the results of this paper have appeared in an earlier paper by the same authors [DDLP05] devoted to the introduction of the notion of *event* bisimulation. As a result of developing the theory of this paper it becomes clear that the points of the state space are less natural than the measurable sets. This suggests working with a "pointless" version of the theory and introducing a Stone-type duality for LMPs. We explore these ideas in a preliminary way in the last section.

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1 Introduction

Markov processes with continuous state spaces or continuous time evolution (or both) arise naturally in several fields of chemistry, physics, biology, economics and computer science. Examples of such systems are Brownian motion, noisy control systems, systems of chemical species undergoing reactions, computer networks, robots and communications systems.

Labelled Markov Processes (LMPs) were formulated [BDEP97, DEP02] to study such general *interacting* Markov processes. In an LMP, instead of one transition probability function (or Markov kernel) there are several, each associated with a distinct label[1]. Each such transition probability function represents the (stochastic) response of the system to an external stimulus represented by the label. In our work we do not associate probabilities with these external stimuli; in other words, we do not intend to quantify the behaviour of the environment. Thus, for those familiar with process algebra terminology, an LMP is a labelled transition system with probabilistic transitions. The interaction is captured by synchronizing on labels in the manner familiar from process algebra.

The following example, taken from [DGJP03], illustrates these ideas.

Example 1.1 Consider the flight management system of an aircraft. It is responsible for monitoring the state of the aircraft – the altitude, windspeed, direction, roll, yaw etc. – periodically (usually several times a second), it also monitors navigational data from satellites and makes corrections, as needed, by issuing commands to the engines and the wing flaps. The physical system is a complex continuous real-time stochastic system; stochastic because the response of the physical system to commands cannot be completely deterministic and also because of unexpected situations like turbulence. From the point of view of the flight management system, however, the system is discrete-time and has continuous space (and one can consider the set of labels corresponding to the commands to be discrete as well). The time unit is the sampling rate. The entire system consists of many interacting concurrent components and programming it correctly – letting alone verifying that the system works – is very challenging. A formal model of this type of software brings us into the realm of process algebra, because of the concurrent interacting components, stochastic processes and real-time systems, the last because the responses may have hard deadlines.

The study of discrete probabilistic systems from the viewpoint of process algebra was initiated by Larsen and Skou [LS91]. They introduced strong probabilistic bisimulation for *discrete* processes in a style similar to the queueing theory notion of "lumpability" invented in the late 1950s [KS60]. In a series of previous papers [BDEP97, DEP98, DEP02] such Markov processes with continuous state spaces and independently acting components were studied, and the phrase "labelled Markov processes" appeared in print explicitly referring to the continuous state space case. Of course, closely related concepts were already around: for example, Markov decision processes [**?**]. The papers by Desharnais, Edalat and Panangaden gave a definition of bisimulation between LMPs, and gave a logical characterization of this bisimulation. Subsequently an approximation theory was developed [DGJP00, DGJP03, **?**] and metrics were defined [DGJP99, **?**, **?**, **?**].

Before we begin the present paper we will briefly review the prior results. The notion of strong probabilistic bisimulation - henceforth just "bisimulation" - was based on the idea that if two states are bisimilar then their transition probabilities to bisimulation equivalence classes should match. This notion works well for the discrete case, but has to be generalised appropriately to the continuous case. One idea was to mimic this definition exactly with a few measure-theoretic

[1] We do not consider *internal* nondeterminism in the present paper.

conditions imposed to deal with the fact that not all sets need be measurable. This was the approach followed in [DGJP00, DGJP03]. However, in an earlier approach [BDEP97, DEP98, DEP02] the authors had defined a bisimulation - they called it a "zigzag" - morphism and then defined a bisimulation relation as a span of such morphisms. This also generalises the discrete case but it turned out to be very painful to prove that one gets a transitive relation and one had to restrict to Polish or analytic spaces. Polish spaces are the topological spaces underlying complete separable metric spaces and as such are very natural: any manifold is a Polish space. However, a quotient construction that we needed does not preserve the Polishness of a space so we had to move to analytic spaces which are, roughly speaking, continuous (or measurable) images of Borel sets in Polish spaces. For analytic spaces (which subsumes Polish) the two notions coincide.

One of the nice things about the theory is the logical characterization of bisimulation. There was already such a theorem for the discrete case in the paper of Larsen and Skou [LS91] but it was not clear that such a theorem would work in the continuous state case. Not only did such a theorem exist but logical characterization worked with a much more parsimonious logic than one was led to expect from the previous work on the discrete case.

There turned out to be a very spartan logic:

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle_q \phi$$

which characterises bisimulation for both continuous and discrete systems. There are two striking things about this logic: *there is no negative construct at all* and one only needs binary conjunction *even though the branching may be uncountable.* The proof heavily uses special properties of analytic spaces that *a priori* have nothing to do with anything logical.

This is an irritating fact: one has to restrict to state-spaces that were analytic. In one sense this is not very restrictive: almost every process that one can imagine has an analytic state space. In particular $\mathbf{R}^n$ with the usual Borel sets is analytic and indeed any manifold is analytic - as we have already remarked. But in another sense it is conceptually unsatisfying that these notions specific to measure theory *on certain topological spaces* should turn out to be so crucial. Why doesn't the theory work for general measure spaces? Certainly the *statement* of the logical characterization theorem does not suggest anything about analytic spaces.

1.1 The road to event bisimulation

The first attempt to define LMPs for continuous systems [BDEP97] did not have any assumptions about the σ-algebra on the state space. LMPs were organized in a category and bisimulation was defined in terms of spans of particular morphisms of this category, called zigzag morphisms. However, buried in the proofs was an alternative view of bisimulation as a cospan: in fact this is the germ of the co-congruence idea that we developed in a previous paper [DDLP05]. With this new definition, one could prove that the logic $\mathcal{L}$ characterizes bisimulation. In the original proof of logical characterization of bisimulation given in [DEP02] this was viewed as an intermediate step. It was not realized that cospans could serve as the defining property of bisimulation. Instead we were obsessed with the idea of bisimulation in terms of spans of morphisms because spans correspond more naturally to *relations* between objects of a category.

For the general theory of bisimulation for processes defined on arbitrary measure spaces, spans could not be used because one could not show that bisimulation was transitive; indeed, this is equivalent to constructing a span given a cospan, and to this day, only analytic spaces have been proven to satisfy this property [Eda99, **?**]. Secondly, the definition of bisimulation was not given on the state-space of a process but, rather, through morphisms in the category: this is not what one was used to working with in the finite case or in the non-probabilistic case. Consequently, a new relational definition of bisimulation (called *state bisimulation* in this paper) was formulated that looked like a nice generalization of non-probabilistic bisimulation as well as of finite probabilistic bisimulation. However, for this definition as well, characterization of bisimulation by the logic was only proven for analytic spaces.

In the paper [DDLP05], we gave a new definition of bisimulation, called *event bisimulation*, that is equivalent to the cospan definition and hence is characterised by the simple logic given above. This is proved without any analyticity assumption: it works for arbitrary measure spaces. Moreover, it has the nice property that the logic yields the biggest possible event bisimulation on any process. We also compare this definition with the state bisimulation mentioned above, and show that the two notions are exactly the same for analytic spaces. More precisely, the largest state bisimulation is an event bisimulation on these spaces. However, viewed as relations we show that roughly speaking, the former is finer that the latter, and hence equates fewer states than the latter in general (in both analytic and non analytic spaces). The following simple example is useful for intuition.

Example 1.2 Consider a set of states equiped with a σ-algebra that does not separate points. For example, one can take $\{1, 2, 3, 4\}$ with the σ-algebra, $\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. The identity relation is a state bisimulation, but it is not an event bisimulation because it distinguishes states that are undistinguishable by the σ-algebra. Indeed states 1 and 2 are not related, whereas no transition probability function may distinguish them, since such functions must respect the σ-algebra in an LMP, by being measurable. Since event bisimulation is based on the σ-algebra, it cannot be finer.

We believe that event bisimulation is the correct generalization of state bisimulation to the category of all LMPs defined for arbitrary measure spaces. This is justified by three arguments: *(i)* the two notions agree on analytic spaces, *(ii)* event bisimulation is shown to be characterised by the logic for any measurable space (analytic and non analytic spaces) and *(iii)* the notion of event bisimulation integrates nicely with the categorical approach to LMPs, and in particular, transitivity of event bisimilarity can be proven categorically without the analyticity assumption.

The use of σ-algebras as means of controlling how much processes know about a particular random outcome is commonplace in probability theory. In that respect, substituting event bisimulation for state bisimulation reconciles the theory of probabilistic processes with traditional probability theory; it allows one to use the σ-algebra as a vector of information to define event bisimulation between processes.

1.2 Outline of the paper

The nice integration of event bisimulation with categorical language (point *(iii)* above) will prove useful for our present purpose which is to enrich the usual notion of LMPs with a notion of negligibles, and carry over the notions of event bisimulation and logical characterization to this enriched framework.

The paper is organised in a way that it is practically self-contained. We start with a brief reminder of the basic concepts and notations which we will need, including a definition of LMPs and the now traditional notion of state bisimulation. Next, we turn to the definition of event bisimulation, because for all the reasons explained above this is the one we want to work with. Together with this definition comes a comparison of the two notions. We show in particular that these notions coincide on analytic state spaces (point *(i)* above). This last point is not needed for the rest of the paper, but by relating our working notion of bisimulation to the traditional one, it helps in building an intuition about the former. As a further step, we recall the LMP modal logic and show the equivalence with event bisimulation (point *(ii)* above). The final preparation step is to recast event bisimulation in categorical terms, that is to say as a co-span of surjective coalgebra morphisms of Giry's monad (point *(iii)* above). This ends the first part of the paper which has essentially appeared in an earlier paper by the same authors [DDLP05].

When this is all in place, we proceed to the second part which constitutes the proper topic of the paper. First, we refine the ambient category of measurable spaces to that of measurable spaces with negligibles. Second, we refine Giry's monad to cope with the newly introduced negligible sets. As a result, we obtain our definition of LMPns and the associated notion of event bisimulation, which parallels the coalgebraic description of LMPs and event bisimulation within our new ambient category and monad. To complete the picture, we turn to the logical side of the matter, and propose a definition of logical equivalence up to negligibles which we prove is indeed a characterization of our refined notion of event bisimilarity.

The introduction of negligibles in the picture promotes a view of MPs where points of the state space are less important, and events become the main characters. A natural step to take then is to develop a pointfree LMP theory and the paper ends with a sketch of what such a theory would look like; this last section also illustrates the algebraic inspiration driving the various choices made in the second part.

1.3 Motivations

At that point, it might be useful to spend some time in motivating the development of a pointfree approach to LMPs. An evident motivation is that there are compelling examples (see the beginning of section 6) of concrete LMPs on continuous spaces which differ only in a point, or in negligibly many points, and which some theory should allow to declare equal, or almost so. When LMPs can be said to be almost equal, and what becomes of the notion of LMP bisimulation in this context, is indeed the question we address in this paper.

A second motivation is that the induced notion of equality in the refined category is rooted in convincing algebraic intuitions, where one sees a measurable function as inducing a morphism between σ-algebras, very much as one sees a continuous function as a frame morphism in pointfree topology. The same can be said of the modified notion of logical equivalence of two systems. The last section of the paper, which by no means constitutes a definitive answer, tries to solidify to some extent these algebraic intuitions by developing rudiments of a Stone duality for LMPs.

One could think of another way to escape from points by switching to a functional description of MPs. Typically, given an MP kernel h, one would define a linear positive functional $F_h(f)(x) = \int f\, dh(x)$ over some space of μ-integrable functions, where μ is some measure. Such function spaces have a natural notion of μ-almost equality attached and are a good starting point for a pointfree perspective. For sure, there are conditions to require for this to make sense, such as the fact that for all x, $h(x)$ should be absolutely continuous with respect to μ. Even more interesting is the fact that functional MPs allow conditional expectation based constructions of approximants [DDP03], where conditioning can be done under arbitrary σ-algebras and not only finite ones, and therefore deliver a more flexible approximation theory for LMPs. On the downside, it seems one has to leave the coalgebraic approach since functional MPs are no longer morphisms in the state space category, and one also has to equip state spaces with somewhat extrinsic measures.

We plan to pursue this functional lead in further work, and compare it with the one we propose here. It is likely that both approaches will end up being close enough, so that conditional expectation based approximation will work to some extent also in the framework which we present here, and this constitutes another if somewhat indirect motivation for our construction.

2 Background

2.1 General background

For S a set and $A \subseteq S$, we write $\mathbf{1}_A$ for A's indicator function. When A, B are disjoint sets, we sometimes write $A + B$ for the disjoint union, and conversely each time we write $A + B$ it is understood that A and B are indeed disjoint. We write $\uparrow A_n$ when A_n is an increasing sequence of sets, that is $A_n \subseteq A_{n+1}$.

Given a set S, one says $\Sigma \subseteq \mathcal{P}(S)$ is a *σ-algebra* over S if (0) Σ contains $\varnothing$, (1) Σ is closed under countable union, and (2) Σ is closed under complement.

This implies that Σ is closed also under countable intersection, and hence under *symmetric difference* defined as $A \triangle B := (A \cap B^c) \cup (A^c \cap B)$.

Examples are the *trivial* σ-algebra $\{\varnothing, S\}$, and the *discrete* one $\mathcal{P}(S)$. Given a subset $\mathcal{C}$ of $\mathcal{P}(S)$, we write $\sigma(\mathcal{C})$ for the smallest σ-algebra containing $\mathcal{C}$, and say that $\mathcal{C}$ generates $\sigma(\mathcal{C})$; then by definition, if $\mathcal{C} \subseteq \Sigma$ and Σ is a σ-algebra, $\sigma(\mathcal{C}) \subseteq \Sigma$.

A *measurable space* is a pair (S, Σ) where S is a set and Σ is a σ-algebra over S. Well-known examples are $[0, 1]$ and $\mathbb{R}$ equipped with their respective *Borel* σ-algebras, written $\mathcal{B}$, generated by intervals.

A subset $\mathcal{N}$ of a σ-algebra Σ is said to be a *σ-ideal* of Σ if (0) $\mathcal{N}$ contains $\varnothing$, (1) $\mathcal{N}$ is a downset of Σ, *i.e.*, if $A \in \mathcal{N}$, $B \in \Sigma$, $B \subseteq A$, then $B \in \mathcal{N}$, and (2) $\mathcal{N}$ is closed under countable union.

Say $A \sim_{\mathcal{N}} B$ if $A \triangle B \in \mathcal{N}$, this is an equivalence relation over Σ (shown later, see subsection 6.1).

Examples of σ-ideals of Σ are $\{\varnothing\}$ and Σ. For a non-trivial example, take the collection of all countable subsets of $[0, 1]$ with the Borel algebra. Given a subset $\mathcal{C}$ of Σ, we write $\mathcal{N}(\mathcal{C})$ for the smallest σ-ideal of Σ containing $\mathcal{C}$, and say that $\mathcal{C}$ generates $\mathcal{N}(\mathcal{C})$.

Suppose Σ is a σ-algebra, $\mathcal{N}$ is a σ-ideal of Σ, and $\Lambda \subseteq \Sigma$ is a sub-σ-algebra of Σ, and define:

$$\Lambda + \mathcal{N} := \{A \triangle N \mid A \in \Lambda,\ N \in \mathcal{N}\}$$

It is easy to see that $\Lambda + \mathcal{N}$ is a sub-σ-algebra of Σ, and the smallest one containing both Λ and $\mathcal{N}$: $\Lambda + \mathcal{N} = \sigma(\Lambda \cup \mathcal{N})$. This can be described equivalently as the closure of Λ under $\sim_{\mathcal{N}}$:

$$\Lambda + \mathcal{N} = \{B \in \Sigma \mid \exists A \in \Lambda : A \sim_{\mathcal{N}} B\}$$

A map f between two measurable spaces (S, Σ) and (S', Σ') is said to be *measurable* if for all $A' \in \Sigma'$, $f^{-1}(A') \in \Sigma$. The smallest σ-algebra such that a given f is measurable is the pointwise inverse image of Σ' and will be denoted by $\sigma(f)$. Hence, f is measurable iff $\sigma(f) \subseteq \Sigma$.

Whether a given f is measurable depends on both σ-algebras Σ and Σ'. For instance, if Σ is the trivial σ-algebra, and Σ' is the discrete one, then f is measurable iff it is constant.

A *subprobability* on (S, Σ) is a map $p : \Sigma \to [0, 1]$, such that for any sequence (A_n) of pairwise disjoint sets in Σ, $p(\sum_n A_n) = \sum_n p(A_n)$. This condition is called σ-additivity; it implies $p(\varnothing) = 0$, as well as a continuity property, namely that for all $\uparrow A_n$ in Σ, $p(\cup A_n) = \sup_n p(A_n)$. If in addition $p(S) = 1$ then one says p is probability. When S is finite and Σ is discrete, a subprobability can be equivalently described as a map p from S to $[0, 1]$ such that $\sum_{s \in S} p(s) \leq 1$.

Given such a probability p, one defines $\mathcal{N}(p) := \{A \in \Sigma \mid p(A) = 0\}$. This is a σ-ideal of Σ. Indeed it is non empty, since $p(\varnothing) = 0$, it is a downset, since if $B \subseteq A$, then $p(B) \leq p(A)$, and it is closed under countable unions. (Indeed, given a sequence $B_n \in \mathcal{N}(p)$, one can define an increasing sequence $A_n = \cup_{i \leq n} B_i$, and then by additivity $p(A_n) \leq \sum_{i \leq n} p(B_i) = 0$, so $p(\cup B_n) = p(\cup A_n) = 0$ by continuity.)

Given two subprobabilities, p, q on a same mesurable space (S, Σ), one says p is absolutely continuous with respect to q, written $p \ll q$, if for all $A \in \Sigma$, whenever $q(A) = 0$ then $p(A) = 0$, or in other words, if $\mathcal{N}(p) \supseteq \mathcal{N}(q)$. When S is finite and Σ discrete, $\mathcal{N}(p)$ is the set of subsets N such that for all $s \in N$, $p(s) = 0$, and $p \ll q$ iff whenever $p(s)$ is 0, so is $q(s)$.

For R a binary relation on a set S, we say that $A \subseteq S$ is *R-closed* if for all $a \in A$, $s \in S$, whenever $(a, s) \in R$, or $(s, a) \in R$, then $s \in A$. For $s \in S$, we write $[s]_R$ for the R-closure of $\{s\}$, which supposing R is an equivalence relation is just the equivalence class of s.

We write $\Upsilon(R)$ for the set of R-closed subsets of S; this is a σ-algebra, since it is clearly closed under any union or intersection, and also closed under complement because our notion of closure is symmetric in R (*i.e.*, a set is R-closed iff it is R^{-1}-closed). We also write $\Sigma(R)$ for $\Sigma \cap \Upsilon(R)$ which is the set of R-closed subsets in Σ, and forms a sub-σ-algebra of Σ.

If $\mathcal{C}$ is a subset of S, we define a binary relation $\mathfrak{R}(\mathcal{C})$ on S by $(s, t) \in \mathfrak{R}(\mathcal{C})$ if for all $A \in \mathcal{C}$, $s \in A \Leftrightarrow t \in A$, or in words if s and t cannot be separated by a set in $\mathcal{C}$. This is an equivalence relation.

2.2 Background on LMPs

Labelled Markov processes are probabilistic versions of labelled transition systems. Corresponding to each label a Markov process is defined. The transition probability is given by a *Markov kernel*. In brief, a labelled Markov process can be described as follows. There is a set of states and a set of labels. The system is in a state at each point in time. The environment selects an action, and the system reacts by moving to another state. The transition to another state is governed by a probabilistic law. For each label there is a transition probability distribution which gives the probability distribution of the possible final states given the initial state. For discrete state spaces, this is essentially the model developed by Larsen and Skou [LS91].

We extended this to continuous-state systems, thus forcing our formalism to be couched in measure-theoretic terms. For instance, we cannot impose requirements on the transition probabilities to arbitrary sets of states — we need to restrict ourselves to measurable sets. The classical theory of Markov processes is typically carried out in the setting of Polish spaces rather than on abstract measure spaces. In previous papers *analytic spaces* - which generalise Polish spaces - were used. However, in this paper we eliminate the need for restricting to analytic spaces.

A key ingredient in the theory is the *Markov kernel*, which we sometimes call a *transition kernel*[2].

[2]In earlier papers, following Feller, we called it a stochastic kernel.

Definition 2.1 *A* ***Markov kernel*** *on a measurable space* (S, Σ) *is a function* $h : S \times \Sigma \to [0, 1]$ *such that for each fixed* $s \in S$*, the set function* $h(s, \cdot)$ *is a subprobability measure, and for each fixed* $X \in \Sigma$ *the function* $h(\cdot, X)$ *is a measurable function.*

One interprets $h(s, X)$ as the probability of the process starting in state s making a transition into one of the states in X. The Markov kernel is really a *conditional probability*: it gives the probability of the process being in one of the states of the set X after the transition, *given* that it was in the state s before the transition.

We will work with transition kernels where $h(s, S) \leq 1$ rather than $h(s, S) = 1$. The mathematical results go through in this extended case. We view processes where the transition kernels are only subprobabilities as being *partially defined*.

Definition 2.2 *A* ***labelled Markov process (LMP)*** $\mathcal{S}$ *with label set* $\mathcal{A}$ *is a structure* $(S, \Sigma, \{h_a | a \in \mathcal{A}\})$*, where* S *is the set of states,* Σ *is a* σ*-algebra on* S*, and for all* $a \in \mathcal{A}$

$$h_a : S \times \Sigma \longrightarrow [0, 1]$$

is a Markov kernel.

We will fix the label set to be $\mathcal{A}$ once and for all, and often forget altogether about actions since it does not restrict the results. In such cases, we will drop the subscript of h. We will use the following notational convention: we write $\mathcal{S}(S, \Sigma, h)$, using the calligraphic font to stand for the LMP and the ordinary capital for the state space.

The all important notion is that of a zigzag morphism.

Definition 2.3 *A* ***zigzag*** *morphism* f *from* $\mathcal{S}$ *to* $\mathcal{S}'$ *is a surjective measurable function* $f : S \to S'$ *satisfying*

$$\forall a \in \mathcal{A}, s \in S, B \in \Sigma' : h_a(s, f^{-1}(B)) = h'_a(f(s), B).$$

We originally defined a bisimulation in terms of spans of zigzags. In order to show transitivity we had to use a subtle construction due to Edalat [Eda99] and later refined and clarified by Doberkat [**?**].

Later we gave the following direct relational definition and finessed the use of Edalat's lemma.

Definition 2.4 *Given an LMP* $\mathcal{S}$*, a* ***(state) bisimulation*** *relation* R *is a binary relation on* S *such that whenever* $(s, t) \in R$ *and* $C \in \Sigma(R)$*, then for all labels* a*,* $h_a(s, C) = h_a(t, C)$*. We say that* s *and* t *are* ***(state) bisimilar*** *if there is any bisimulation* R *such that* $(s, t) \in R$*.*

Note that the reflexive, symmetric, and transitive closure of any binary R relation satisfying the condition of Definition 2.4, also satisfies this condition (because $\Sigma(R)$ is invariant under this closure). We may then assume for the rest of the paper that state bisimulations are equivalence relations. This will ease the comparison with event bisimulation (to be defined in the next section).

One can define a simple modal logic called $\mathcal{L}$ which has the following syntax:

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle_r \phi$$

where a is an action and r is a rational number between 0 and 1. The last formula is interpreted as follows. We say $s \models \langle a \rangle_q \phi$ if s can make an a-transition with probability greater than or equal to the rational number r and end up in a state satisfying ϕ.

Despite the fact that $\mathcal{L}$ is a very parsimonious logic, since it has no negation and no infinitary constructs, one can establish that two states are (state) bisimilar if and only if they satisfy exactly the same formulas. Indeed for finite-state processes one can decide whether two states are bisimilar and effectively construct a distinguishing formula in case they are not [DGJP02]. This theorem is refered to as the *logical characterization* theorem, and, for (state) bisimulation, it works only when the state space is an analytic space.

In a later section we will show that the new notion of bisimulation leads to a new logical characterization theorem for LMPs where the analyticity condition is no longer needed. That is to say the new characterization theorem will be seen to hold for arbitrary measurable spaces. Since both notions of bisimulations agree on analytic spaces (this is proved in the next section), one sees that the new characterization theorem extends the older one.

3 Event and state bisimulation

In defining a notion of probabilistic bisimulation one is forced to add transition probabilities. It makes no sense to compare labels and transition probabilities between individual transitions. If one were to do so then obvious examples would fail to be bisimilar. Consider a three-state system with a-transitions from state 1 to states 2 and 3, each with a probability $\frac{1}{2}$, and compare it with a two-state system with a probability 1 a-transition from state 1 to state 2. Clearly we need to aggregate the transition probabilities in the first system in order to realize that the two systems are really bisimilar. The question is how should one aggregate?

Equivalence relations are the first thing that leaps to mind and this leads to state bisimulation [DGJP03]. One can think of relations as spans of appropriate functions; in this case we may use the zigzags [DEP02] described in the last section. On the other hand there is a natural family of subsets at hand: the measurable sets. One can think of the σ-algebra and sub-σ-algebras as defining families of interesting sets of states. It is curious that measurability does not crop up as a restriction on the possible bisimulation relations. To be sure, one has to look at R-closed measurable sets but there is no restriction such as that R should be an equivalence relation with measurable equivalence classes (it would be hard to prove transitivity if measurability was imposed on R). The notion of event bisimulation puts measurability front and centre.

It is helpful to reformulate state bisimulation in order to facilitate the switch to event bisimulation.

Lemma 3.1 *Given an LMP (S, Σ, h), R is a state bisimulation iff $(S, \Sigma(R), h)$ is an LMP.*

Proof: Let A be in Σ, and a be an action, we show first that $h_a(\cdot, A)$ is $\Sigma(R)$-measurable iff it is constant on R-classes.
$\Leftarrow$: Since $h_a(\cdot, A)$ is Σ-measurable, $\langle a\rangle_r(A) := \{s \mid h(s, A) \geq r\} = h_a(\cdot, A)^{-1}[r, 1]$ is in Σ, and since $h_a(\cdot, A)$ is constant on R-classes, $\langle a\rangle_r(A)$ is also R-closed. Hence $\langle a\rangle_r(A) \in \Sigma(R)$, for all r, which is equivalent to saying that $h_a(\cdot, A)$ is $\Sigma(R)$-measurable.
$\Rightarrow$: Since $h_a(\cdot, A)$ is $\Sigma(R)$-measurable, for every $r \in [0, 1]$, $\langle a\rangle_r(A)$ is in $\Sigma(R)$, and therefore R-closed, which is equivalent to saying that $h_a(\cdot, A)$ is constant on R-classes.

By definition, R is a state bisimulation iff for all A in $\Sigma(R)$, $a \in \mathcal{A}$, $h_a(\cdot, A)$ is constant on R-classes, which by the argument above is iff $h_a(\cdot, A)$ is $\Sigma(R)$-measurable, which by definition is iff $(S, \Sigma(R), h)$ is an LMP. ∎

This shows that one can work with a smaller σ-algebra closely connected with bisimulation equivalence, but note that different state bisimulations can yield the same σ-algebra. In fact one has a new LMP with the same kernel and state space but with a reduced σ-algebra. Note that if R is the identity relation, then $\Sigma(R) = \Sigma$. It is easy to see that

Lemma 3.2 *R is a state bisimulation iff the identity map $i : (S, \Sigma, h) \to (S, \Sigma(R), h)$ is a zigzag.*

Ideally one would like it to be the case that any zigzag induces a state bisimulation on its domain. However this is false. Given a zigzag $f : (S, \Sigma, h) \to (S', \Sigma', h')$ and R the relation induced on S by f we can conclude that if $f(s_1) = f(s_2)$ then $h(s_1, A) = h(s_2, A)$ for every A in $f^{-1}(\Sigma') \subseteq \Sigma(R)$ and not for every A in $\Sigma(R)$. Thus the equivalence induced by a zigzag is too fine in general. The crucial point is that we are making the equivalence relation primary and the σ-algebra secondary. Instead we should work with the structure naturally associated with a σ-algebra: in other words we should look for a sub-σ-algebra.

Definition 3.3 *An **event bisimulation** on an LMP (S, Σ, h) is a sub-σ-algebra Λ of Σ such that (S, Λ, h) is an LMP.*

We will refer to both Λ and its associated equivalence $\mathfrak{R}(\Lambda)$ as event bisimulation.

The phrase "event bisimulation" is meant to suggest that the focus of interest has shifted from the individual points of S to the measurable sets, or - to emphasize the probabilistic interpretation - the events. What has happened here is that the notion of bisimulation qua relation has been replaced by an arbitrary sub-σ-algebra rather than $\Sigma(R)$, the sub-σ-algebra generated by a relation R. The key point is that the transition kernels have to have the appropriate measurability properties with respect to Λ. We can more sensibly say that Λ, rather than $\mathfrak{R}(\Lambda)$, is an event bisimulation.

The similarity with state bisimulation can be made quite striking.

Lemma 3.4 *If Λ is an event bisimulation, then the identity function on S defines a zigzag morphism from (S, Σ, h) to (S, Λ, h).*

If we compare this result to Lemma 3.2, we can see that it fits more nicely in the category of LMPs in that it does not talk about relations. Moreover, we get a perfect correspondence with zigzags.

Lemma 3.5 *If $f : (S, \Sigma, h) \to (S', \Sigma', h')$ is a zigzag morphism, then $f^{-1}(\Sigma')$ is an event bisimulation on S.*

In order to facilitate the study of the relation between state and event bisimulation, we need some elementary mathematical observations connecting σ-algebras and binary relations.

Given a measurable space (S, Σ), we have two maps back and forth between sub-σ-algebras of Σ and equivalence relations over S: $\Lambda \mapsto \mathfrak{R}(\Lambda)$ and $R \mapsto \Sigma(R)$. Note that these maps are antimonotone, indeed if $\Lambda \subseteq \Lambda'$, then $\mathfrak{R}(\Lambda) \supseteq \mathfrak{R}(\Lambda')$ because Λ being smaller it separates less points, and conversely $R \subseteq R'$, then $\Sigma(R) \supseteq \Sigma(R')$, since R'-closed subsets are a fortiori R-closed.

Lemma 3.6 *Let (S, Σ) be a measurable space, R a relation on S and $\Lambda \subseteq \Sigma$ a sub-σ-algebra:*
(i) $\Lambda \subseteq \Sigma(\mathfrak{R}(\Lambda))$.
(ii) $R \subseteq \mathfrak{R}(\Sigma(R))$.
(iii) If R-equivalence classes are in Σ, then $R = \mathfrak{R}(\Sigma(R))$.

Proof . (i) Suppose A is in Λ, then $A \in \Sigma$, and also A is $\mathfrak{R}(\Lambda)$-closed (since if there is $x \in A$, $y \notin A$, then $y \in A^c \in \Lambda$, so (x, y) is not in $\mathfrak{R}(\Lambda)$). Thus $A \in \Sigma \cap \Upsilon(\mathfrak{R}(\Lambda)) = \Sigma(\mathfrak{R}(\Lambda))$. (ii) Next, we have $[s]_{\mathfrak{R}(\Sigma(R))} = \cap A \ni s$ with $A \in \Sigma$ and R-closed; and $[s]_R = \cap B \ni s$ with B R-closed; so $[s]_R \subseteq [s]_{\mathfrak{R}(\Sigma(R))}$, and the conclusion follows. Now for $\supseteq$ in (iii), if $[s]_R$ itself is in Σ then it qualifies as an A in the big intersection above and the result follows. ∎

Proposition 3.7 *Let (S, Σ) be a measurable space and $\Lambda \subseteq \Sigma$ a sub-σ-algebra:*
(i) $\mathfrak{R}(\Lambda) = \mathfrak{R}(\Upsilon(\mathfrak{R}(\Lambda)))$.
(ii) If $\Lambda \subseteq \Sigma$ then $\mathfrak{R}(\Lambda) = \mathfrak{R}(\Sigma(\mathfrak{R}(\Lambda)))$.

Proof . (i) Set $R := \mathfrak{R}(\Lambda)$, $\Sigma := \Upsilon(\mathfrak{R}(\Lambda))$, obviously one has that R-classes are in Σ, so (by Lemma 3.6 (iii)) $R = \mathfrak{R}(\Sigma(R))$, on the other hand $\Sigma(R) = \Sigma$, and the conclusion follows. (ii) One has $\Lambda \subseteq \Sigma(\mathfrak{R}(\Lambda)) := \Sigma \cap \Upsilon(\mathfrak{R}(\Lambda)) \subseteq \Upsilon(\mathfrak{R}(\Lambda))$, so by antimonotony $\mathfrak{R}(\Lambda) \supseteq \mathfrak{R}(\Sigma(\mathfrak{R}(\Lambda))) \supseteq \mathfrak{R}(\Upsilon(\mathfrak{R}(\Lambda))) = \mathfrak{R}(\Lambda)$ by (i). ∎

These lemma and proposition show how to transfer results between sub-σ-algebras and equivalence relations.

We presented event bisimulation as a weakening of state bisimulation. Consequently, we would like to prove that the latter is always an event bisimulation. However, this is not actually the case in general: essentially because there is not enough measure theoretic control on the relations used to define state bisimulation.

We know from Lemma 3.6 that $R \subseteq \mathfrak{R}(\Sigma(R))$. The following lemma shows that if a state bisimulation satisfies the reverse inclusion, then it is an event bisimulation.

Lemma 3.8 *Let R be a state bisimulation over (S, Σ, h), then R is an event bisimulation if and only if $R = \mathfrak{R}(\Sigma(R))$.*

Proof . $\Rightarrow$: If R is an event bisimulation, then by definition $R = \mathfrak{R}(\Lambda)$ for some sub-σ-algebra Λ of Σ. We have that $\Lambda \subseteq \Sigma(R)$ because every $A \in \Lambda$ is evidently $\mathfrak{R}(\Lambda)$-closed. Thus $R = \mathfrak{R}(\Lambda) \supseteq \mathfrak{R}(\Sigma(R))$, and the reverse inclusion is given by Lemma 3.6 (ii).
$\Leftarrow$: If R is a state bisimulation we know that $(S, \Sigma(R), h)$ is an LMP by Lemma 3.1, so R is an event bisimulation for $\Lambda = \Sigma(R)$, since by assumption $\mathfrak{R}(\Sigma(R)) = R$. ∎

Putting together this result and Lemma 3.6 (iii), we obtain:

Corollary 3.9 *If relation R is a state bisimulation with equivalence classes in Σ, then R is an event bisimulation.*

This immediately implies that a state bisimulation is an event bisimulation in the case of a countable space with every set being measurable; because in this case any equivalence relation will have measurable classes.

Corollary 3.10 *If $\mathcal{S}$ is countable with discrete σ-alegbra, any state bisimulation is an event bisimulation.*

The condition that $\Sigma = \mathcal{P}(S)$ is necessary in this corollary as one can see from Example 1.2. Note that this is a necessary and sufficient condition for a countable measurable space to be analytic. This represents the essential difference between analytic and non analytic state spaces for discrete LMPs.

We cannot expect that every state bisimulation satisfies the equality in Lemma 3.8, as the following example shows.

Example 3.11 Clearly there exists equivalence relations R such that $R \subset \mathfrak{R}(\Sigma(R))$, even on the analytic space $(\mathbb{R}, \mathcal{B})$. Pick $V \subseteq \mathbb{R}$ not in $\mathcal{B}$, and define R by the two equivalence classes V and V^c. Since $\Sigma(R)$ is the trivial σ-algebra, $\mathfrak{R}(\Sigma(R))$ equates everything and therefore is different from R.

We may even define an LMP h over $(\mathbb{R}, \mathcal{B})$ such that R is a state bisimulation and yet $R = \mathfrak{R}(\Lambda)$ for no event bisimulation Λ. First we remark that if $R = \mathfrak{R}(\Lambda)$, then Λ cannot separate V, so the only possibility is $\Lambda = \{\varnothing, V, V^c, \mathbb{R}\}$. Pick s, $s' \in V$ distinct (V not being a Borelian is necessarily uncountable), and define h such that $h(s, \{s\}) = 1$ and $h(s', \{t\}) = 1$ for some $t \notin V$; R is a state bisimulation because the only non-empty measurable in $\Sigma(R)$ is $\mathbb{R}$, but as follows also from Lemma 3.8, R is not an event bisimulation for Λ because the kernel h is not Λ-measurable.

The conclusion is that, even on an analytic space, not every state bisimulation is an event bisimulation. This may seem to kill any hope to show that a state bisimulation is an event bisimulation. However, one can observe that when this is the case, the state bisimulation distinguishes too many states, in particular it distinguishes states that are not separable by Σ. This is a liberty that an event bisimulation never has. The following result shows that assuming the fact that "a bigger state bisimulation is better", we do have that result.

Proposition 3.12 *If R is a state bisimulation, then $\mathfrak{R}(\Sigma(R))$ is a state bisimulation and an event bisimulation.*

Proof . By Lemma 3.1, R is a state bisimulation iff $\Sigma(R)$ is an event bisimulation, that is iff $R(\Sigma(R))$ is an event bisimulation. It remains to show that $R(\Sigma(R))$ is also a state bisimulation. Again by lemma 3.1 this amounts to saying that $\Sigma(\mathfrak{R}(\Sigma(R)))$ is an event bisimulation, but by Lemma 3.13 below, $\Sigma(\mathfrak{R}(\Sigma(R)))$ is no other than $\Sigma(R)$, which we have shown to be an event bisimulation. ∎

Lemma 3.13 *If R is a state bisimulation, then $\Sigma(R) = \Sigma(\mathfrak{R}(\Sigma(R)))$.*

Proof . $\Sigma(R) \supseteq \Sigma(\mathfrak{R}(\Sigma(R)))$ because $R \subseteq \mathfrak{R}(\Sigma(R))$. For inclusion, let $X \in \Sigma(R)$, then X is $\mathfrak{R}(\Sigma(R))$-closed and in Σ. Thus it is in $\Sigma(\mathfrak{R}(\Sigma(R)))$. ∎

Proposition 3.12 implies that if a state bisimulation is not an event bisimulation, then it can be expanded to one that is. One example is the identity relation, which is not an event bisimulation when Σ does not separate points, but it is a state bisimulation. This is an example of a relation that sees more differences than Σ can see.

Lemma 3.14 *The identity relation I is a state bisimulation; it is an event bisimulation iff Σ separates points.*

Proof . $\Sigma(I) = \Sigma$ and lemma 3.1 proves the first point. For the second one, suppose $I = \mathfrak{R}(\Lambda)$ for some $\Lambda \subseteq \Sigma$; then Λ separates points in S, so Σ also does; in fact $I = \mathfrak{R}(\Sigma)$ in this case.

Conversely if Σ separates points, then $I = \mathfrak{R}(\Sigma)$ and obviously (S, Σ, h) is an LMP, so I is an event bisimulation. ∎

One interpretation of this is that the correct "identity relation" in measure spaces is the one generated by the σ-algebra rather than the usual one defined on the points.

An important remark has to be made. We know from Proposition 3.12 that the largest state bisimulation is an event bisimulation. We will see in the next section that the largest event bisimulation is also a state bisimulation for analytic spaces. We do not know if it is the case for non analytic spaces. The question is equivalent to asking if state bisimulation is characterised by the logic in any LMP. Indeed, we know from previous work [DEP98, DEP02] that state bisimulation is characterised by the logic for analytic spaces, and we will prove in Section 5 that event bisimulation is also, for any LMP.

3.1 Analytic spaces

We have already shown that for countable spaces with discrete σ-algebras the two notions, state and event bisimulation coincide. In fact they coincide for the vastly larger class of analytic spaces. Indeed the proofs of logical characterization of bisimulation given in previous papers essentially establish this fact though it is hidden in the proofs. They hinge on the special properties of countably generated σ-algebras. The following lemma from [**?**] uses the unique structure theorem of analytic spaces, and is all we need to know about these spaces in the context of this paper. Recall that, if $\mathcal{C} \subseteq \Sigma$, we write $\sigma(\mathcal{C})$ for the smallest σ-algebra containing $\mathcal{C}$.

Lemma 3.15 *Let (S, Σ) be an analytic space, and $\mathcal{C} \subseteq \Sigma$ be countable, with $S \in \mathcal{C}$, then $\Sigma(\mathfrak{R}(\mathcal{C})) = \sigma(\mathcal{C})$.*

Lemma 3.16 *If $\mathcal{S}(S, \Sigma, h)$ be an LMP with (S, Σ) an analytic space, and Λ be a countably generated event bisimulation, then $\mathfrak{R}(\Lambda)$ is a state bisimulation.*

Proof . Pick $\mathcal{C}$ countable, containing S, and generating Λ. One has that $\mathfrak{R}(\mathcal{C}) = \mathfrak{R}(\Lambda)$ (suppose s, t are not separated within some $\mathcal{D}$, then they are not separated either by complements of sets in $\mathcal{D}$, nor by arbitrary unions, so $\mathfrak{R}(\mathcal{D}) = \mathfrak{R}(\sigma(\mathcal{D}))$) so by Lemma 3.15, one also has $\Sigma(\mathfrak{R}(\Lambda)) = \sigma(\mathcal{C}) = \Lambda$, so that $\Sigma(\mathfrak{R}(\Lambda))$ is an event bisimulation, and by Lemma 3.1, $\mathfrak{R}(\Lambda)$ is a state bisimulation. ∎

Proposition 3.17 *An LMP (S, Σ, h) always has a maximal state bisimulation R, and a minimal event bisimulation Λ; if furthermore (S, Σ) is analytic, then $R = \mathfrak{R}(\Lambda)$.*

Proof . There is always a minimal event bisimulation which is just the intersection of all event bisimulations. Likewise, there is always a maximal state bisimulation which is the union of all state bisimulations, because state bisimulations are closed under arbitrary unions. To see this, let $R = (\cup_i R_i)$, and notice first that $\Sigma(R) = \cap_i \Upsilon(R_i) \cap \Sigma$, indeed a set is R-closed iff for all i it is R_i-closed. Now, R is a state bisimulation iff $\Sigma(R)$ is an event bisimulation iff for all $B \in \Sigma(R)$, $a \in \mathcal{A}$, $r \in [0, 1]$, one has $\langle a \rangle_r(B) \in \Sigma(R)$. Since $\Sigma(R) \subseteq \Sigma(R_i)$, and R_i is a state bisimulation,

$\langle a\rangle_r(B) \in \Sigma(R_i)$, for all i, so is in $\Sigma(R)$. Hence, $\Sigma(R)$ is stable under the $\langle a\rangle_r$, and $\Sigma(R)$ is an event bisimulation, and, by Lemma 3.1 and Definition 3.3, R is therefore a state bisimulation.

As we will see in the next section (Proposition 4.5), the minimal event bisimulation Λ of any LMP is countably generated. Since we suppose (S, Σ) analytic, $\mathfrak{R}(\Lambda)$ is also a state bisimulation (by Lemma 3.16). To show it is maximal, pick another state bisimulation R, one has $\Lambda \subseteq \Sigma(R)$ because Λ is minimal, hence $\mathfrak{R}(\Lambda) \supseteq \mathfrak{R}(\Sigma(R)) \supseteq R$. ∎

Moreover, Lemma 3.16 implies Corollary 3.10 and its counterpart:

Corollary 3.18 *If $\mathcal{S}$ is countable with discrete σ-algebra, state bisimulation is exactly event bisimulation.*

Proof . Let Λ be an event bisimulation over S, it is obviously countably generated by the equivalence classes because S is countable. Since a countable space with discrete σ-algebra is analytic, we have that $\mathfrak{R}(\Lambda)$ is a state bisimulation (by Lemma 3.16). ∎

4 Logical characterization

The logical characterization of probabilistic bisimulation [DEP98, DEP02] as it was originally proved worked with what we are now calling state bisimulation and was established for analytic spaces. In the present section we establish the logical characterization of bisimulation for event bisimulation for LMPs defined on general measure spaces. In conjunction with the results of the previous section - that the two notions are essentially the same on analytic spaces - it implies the earlier logical characterization of bisimulation result. Moreover, it shows that the role of analytic spaces can be confined to a single lemma, namely Lemma 3.15. At the end of the section we explicate the measure theoretic significance of the particular logic $\mathcal{L}$ that we used.

We recall the logic

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a\rangle_q \phi.$$

Given a formula ϕ we write $[\![\phi]\!]$ for the set of states satisfying the formula ϕ. It is easy to see that these are all measurable sets. We write $[\![\mathcal{L}]\!]$ for the collection of sets of the form $[\![\phi]\!]$. The key point that we shall establish is that the σ-algebra generated by the logic is an event bisimulation; moreover it is the maximal event bisimulation. From this the logical characterization of event bisimulation is immediate.

The proofs depend on properties of π-systems and d-systems. We recall the basic definitions from the literature [Wil91].

Definition 4.1 *Let S be a set: (1) a π-system on S is a subset of $\mathcal{P}(S)$ closed under (finite) intersection and containing S, (2) a d-system $\mathcal{D}$ on S is a subset of $\mathcal{P}(S)$ containing S, closed under countable increasing unions and relative complements (i.e., if $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$).*

The point of π-systems is that one can often work with them instead of the σ-algebras that they generate; usually the sets of a π-system are much simpler than the sets in the generated σ-algebra.

Another key concept, and the one that brings out the special role of the logic $\mathcal{L}$ is stability.

Definition 4.2 *Let (S, Σ, h) be an LMP and $\mathcal{C} \subseteq \Sigma$, we say that $\mathcal{C}$ is **stable** with respect to (S, Σ, h) if for all $A \in \mathcal{C}$, $r \in [0, 1]$, $a \in \mathcal{A}$,*

$$\langle a \rangle_r A := \{s \mid h_a(s, A) > r\} \in \mathcal{C}$$

Clearly both $\langle a \rangle_r A$ is measurable if A is and h is a Markov kernel. Note also that Λ is an event bisimulation on (S, Σ, h) if and only if it is a stable sub-σ-algebra of Σ, and that the condition of measurability of a kernel $h(\cdot, A)$ is exactly that Σ be stable. Sometimes we write simply $\langle\rangle_r(A)$ when the actions are not taken into account (as will be often the case in the remainder of the paper).

Proposition 4.3 *$[\![\mathcal{L}]\!]$ is the smallest stable π-system of (S, Σ, h).*

Proof . Evident. ∎

Lemma 4.4 *If $\mathcal{C}$ is a stable π-system of (S, Σ, h), then $\sigma(\mathcal{C})$ is also stable.*

Proof . We show that $\mathcal{D} = \{A \in \Sigma \mid \forall a \forall q \, \langle a \rangle_q(A) \in \sigma(\mathcal{C})\}$ is a d-system.
(i) $S \in \mathcal{D}$ because $S \in \mathcal{C}$ and $\mathcal{C}$ is stable;
(ii) if A, $B \in \mathcal{D}$ and $A \subset B$, then

$$\{s \mid h_a(s, B \setminus A) < q\} = \cup_{r \geq q} (\langle a \rangle_r(B))^c \cap (\langle a \rangle_{r-q}(A)),$$

because $h_a(s, B \setminus A) < q$ iff there exists r such that $h_a(s, B) \leq r$ and $h_a(s, A) > r - q$, where r can always be chosen rational, so we have a countable union of measurable sets and the lhs is in $\sigma(\mathcal{C})$, and so is clearly $\langle a \rangle_q(B \setminus A)$, implying that $B \setminus A \in \mathcal{D}$;
(iii) if $A_n \in \mathcal{D}$ and $A_n \uparrow$, then $\langle a \rangle_r(\cup_n A_n) = \cup_n \langle a \rangle_r(A_n)$, because $h_a(s, \cup_n A) = \sup_n h_a(s, A_n)$ (continuity property for increasing unions), and hence $\cup_n A_n \in \mathcal{D}$.

This shows that $\mathcal{D}$ is a d-system, and since $\mathcal{C} \subset \mathcal{D}$, because $\mathcal{C}$ is stable, by the monotone class theorem [Wil91], we have that $\sigma(\mathcal{C}) \subset \mathcal{D}$. In other words, $\sigma(\mathcal{C})$ is stable. ∎

This gives us characterization of event bisimulation by the logic $\mathcal{L}$.

Proposition 4.5 *$\sigma([\![\mathcal{L}]\!])$ is the smallest stable σ-algebra included in Σ.*

Proof . Let Σ_m be the smallest stable σ-algebra included in Σ.
By Lemma 4.3, $[\![\mathcal{L}]\!] \subseteq \Sigma_m$, because Σ_m is a stable π-system, and hence $\sigma([\![\mathcal{L}]\!]) \subseteq \Sigma_m$.
Conversely, $[\![\mathcal{L}]\!]$ is a stable π-system by Lemma 4.3, and hence, by Lemma 4.4, $\sigma([\![\mathcal{L}]\!])$ is stable and hence contains Σ_m. ∎

Corollary 4.6 *The logic $\mathcal{L}$ characterizes event bisimilarity.*

Proof . From Proposition 4.5, stability tells us that $\sigma([\![\mathcal{L}]\!])$ is an event bisimulation and the fact that it is the smallest implies that any event bisimulation preserves $\mathcal{L}$ formulas. ∎

Moreover, Proposition 4.5 yields an interesting definition of $\mathcal{L}$ on a pure measurability basis, by providing a countable set of generators for the smallest event bisimulation, which we used in the course of the proof of Proposition 3.17.

5 Event bisimulation as probabilistic cocongruence

Whether one wants to specify a process, or to prove some relevant properties - perhaps by automated methods as in model-checking - transition systems offer a convenient description language which is perhaps the single most widely used model in computer science.

Depending on the particulars of the process of interest, the associated transition systems may have to incorporate various computational ingredients. As explained in the introduction, one may want to include labels to model the interaction of the process with its context or environment, and to give means to describe a whole system out of simpler components. One may also want to include probabilities, giving a quantitative measure of the likelihood of events related to the evolution of the process, as well as non-deterministic behaviour, when quantitative information is not available or meaningful.

This diversity of computational features has been a strong incentive to recast transition systems into categorical language and specifically to obtain them as coalgebras of certain functors. In this section we recall this categorical - more precisely, coalgebraic - treatment of LMPs [Gir81, dVR97, **?**] and event bisimulation [DDLP05]. This will prepare the ground for our refinement of the notion of LMP in the next section.

For simplicity we will elide labels. In talking about bisimulation categorically it makes more sense to think of it as a relation between different LMPs rather than as a relation on the state space of a single LMP. This is a very slight shift in point of view. At the beginning we will mention the connection but as the discussion proceeds we will just talk about state and event bisimulation between different LMPs.

We start with a reminder of the usual coalgebraic construction of LMP.

5.1 The category Mes and Giry's monad Π

Objects in **Mes** are measurable spaces (S, Σ), and arrows are measurable functions. This data clearly defines a category. This category has pullbacks and finite products constructed just as in **Set**. More importantly in the context of this paper, it has coequalizers and finite coproducts - also constructed just as in **Set** - and hence, all finite colimits; in particular, it has pushouts. These are useful in the study of bisimulation.

Giry [Gir81] defines an endofunctor Π on **Mes** taking (S, Σ) to

$$\Pi S \quad := \quad \{\nu \mid \Sigma \to [0,1] \mid \nu \text{ is a (sub)probability on } (S, \Sigma)\}.$$

For $A \in \Sigma$, the *evaluation map* $e_A : \Pi S \to ([0,1], \mathcal{B})$ is given as $e_A(\nu) = \nu(A)$. To complete the definition of ΠS as an object, one equips it with the smallest σ-algebra such that all evaluation maps e_A are measurable. Equivalently:

$$\Sigma_{\Pi S} \quad := \quad \sigma\{e_A^{-1}(\alpha, 1];\ \alpha \in [0,1],\ A \in \Sigma\}$$

The arrow part of the endofunctor Π is $\Pi(f)(\nu) := \nu \circ f^{-1}$. This is the familiar image measure construction, and it is clearly functorial in f. Just as clearly this map $\Pi(f)$ is measurable since $e_A(\Pi(f)(\nu)) > \alpha$ iff $\nu(f^{-1}(A)) > \alpha$ iff $e_{f^{-1}(A)}(\nu) > \alpha$.

Labelled Markov processes are just coalgebras of Π, that is to say morphisms of the form $S \to \Pi S$ for some object S in **Mes**. Indeed, a measurable function $h : S \to \Pi S$ is - if curried appropriately - exactly a Markov kernel as in Definition 2.1. Indeed, such an h is a morphism, by

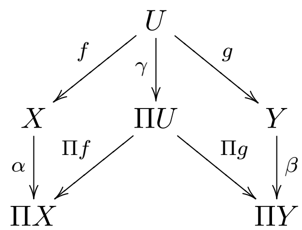


Figure 1: Bisimulation as a span in a coalgebra category.

definition of $\Sigma_{\Pi S}$, iff for all $A \in \Sigma_S$, $\alpha \in [0,1]$, $h^{-1} \circ e_A^{-1}(\alpha, 1] \in \Sigma_S$, which is iff $h(., A)^{-1}(\alpha, 1] \in \Sigma_S$ is measurable (modulo currying).

The category of coalgebras of Π has for objects Π-coalgebras. A morphism from $h : S \to \Pi S$ to $k : T \to \Pi T$ is a morphism $f : S \to T$ in **Mes**, such that $kf = \Pi(f)h$. It is easy to see that a coalgebra morphism is precisely a zigzag morphism minus the surjectivity condition. This coalgebraic construction and its relation to bisimulation were developed by de Vink and Rutten [dVR97, **?**] and noted in passing in [BDEP97].

Moreover, a multiplication map $\mu_S : \Pi^2 S \to \Pi S$ is given by

$$\mu(\Omega)(A \in \Sigma) \quad = \quad \int_{\Pi S} e_A d\Omega$$

and a unit map $\eta_S : S \to \Pi S$ is $\eta(x) = \delta_x$, the Dirac measure concentrated at x. Together with the unit and multiplication maps, $\mathbf{\Pi}$ constitutes a monad, so that LMPs can be composed, and can be seen as morphisms of the so-called Kleisli category associated to $\mathbf{\Pi}$.

Incidentally, MPs and variations thereof can also be presented via distributive laws between various monads [Bur03].

5.2 Bisimulation as cospans

Usually bisimulation is defined as a span in the coalgebra category as shown in Fig. 1. Here α and β define Π-coalgebras on X and Y respectively; in other words they define LMPs. The span of zigzags given by f and g - with f and g both surjective - defines a bisimulation relation between (X, α) and (Y, β).

Any bisimulation defined in this way also produces a state bisimulation. Specifically, suppose $f, g : \mathcal{T} \to \mathcal{S}$ is a pair of zigzags, and write R for the associated relation over S, defined as $(s, s') \in R$ if $s = f(t)$ and $g(t) = s'$ for some t. One has that if $(s, s') \in R$ and $B \in \Sigma_S$ is R-closed:

$$h_S(s, B) = h_T(t, f^{-1}B) = h_T(t, g^{-1}B) = h_S(s', B)$$

where the second step is because if B is R-closed then $f^{-1}B = g^{-1}B$, supposing R to be symmetric as we do. So one obtains indeed a state bisimulation from a (symmetric) span. However, note that the converse is not clear; *i.e.*, it is not clear that given a state bisimulation over $\mathcal{S}$, one can define a span of zigzags over $\mathcal{S}$ such that the associated state bisimulation is the one one started with.

Let us think nevertheless of bisimulation as a span between two LMPs. One needs to show that bisimulation is transitive. That is, given spans from $\mathcal{S}_1$ to $\mathcal{S}_2$ and from $\mathcal{S}_2$ to $\mathcal{S}_3$ we would like to

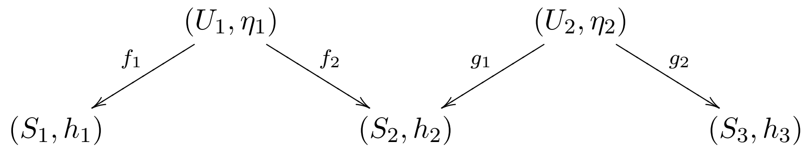


Figure 2: Composing spans

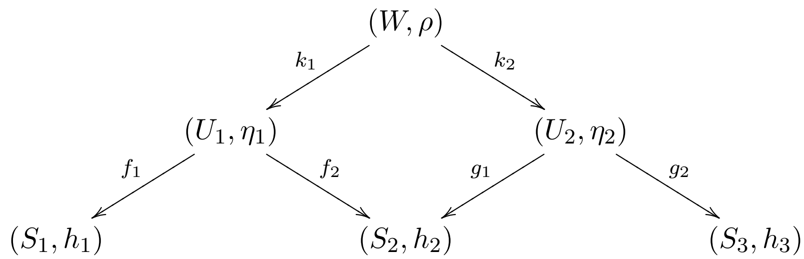


Figure 3: Composing spans given weak pullbacks.

construct a span from $\mathcal{S}_1$ to $\mathcal{S}_3$. Given this situation we have a co-span formed by $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{S}_2$ and the zigzags f_2 and g_1 as shown in Fig. 2.

Usually one postulates the existence of pullbacks, or at least weak pullbacks, in order to complete the square as shown in Fig. 3.

However, these weak pullbacks need to exist in the category of coalgebras, not just in the category **Mes**. In order for this to happen one needs to have that Π preserves weak pullbacks. This rarely happens. In the original de Vink and Rutten papers [dVR97, **?**] the existence of weak pullbacks in the category of coalgebras was shown in the discrete setting. Edalat [Eda99] produced a much weaker construction - he called it a "semi-pullback" - which allows one to complete the square, but has no universal properties, in the category of coalgebras of Π over the base category of analytic spaces equipped with their Borel σ-algebra. The semi-pullback construction shows immediately that the span definition of bisimulation [BDEP97, DEP02] is transitive.

With cospans everything works much more smoothly. In fact, cospans are the natural structure to use if one is interested in equivalence relations. To begin, we observe that if we work with cospans then we can compose using pushouts and *this does not require* Π *to preserve anything.* We consider the situation shown in Fig. 4 where we have omitted labels for some of the arrows, for example $\Pi f : \Pi X \to \Pi U$, where they can be inferred by functoriality. The arrows f, g, f' and g' are zigzags.

One way to construct a cospan from $\mathcal{X}$ to $\mathcal{Z}$ is to construct a pushout in the coalgebra category. In **Mes** we can construct a pushout for the arrows g and f to obtain the situation shown in Fig. 5. Here W is the object constructed as the tip of the pushout in **Mes**. In order to have a pushout in the category of coalgebras we need to put a coalgebra structure on W, *i.e.*, we need to construct a morphism $\rho : W \to \Pi W$, shown dotted in the diagram. Consider the following calculation:

$$g; h; \Pi i$$

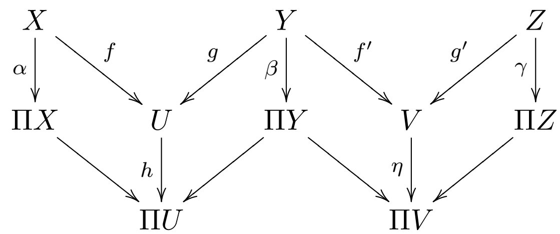


Figure 4: Composing cospans.

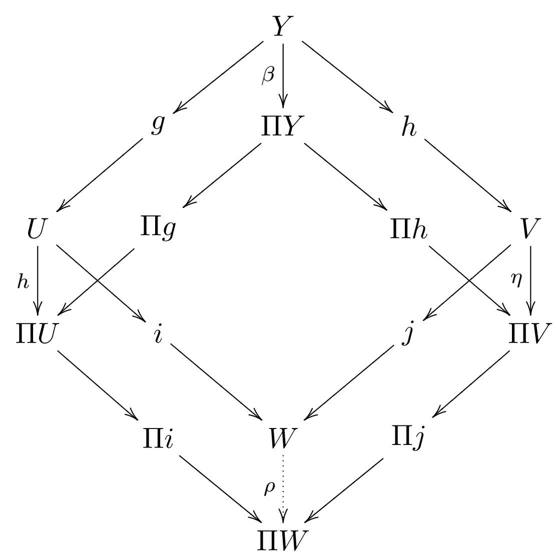


Figure 5: Pushouts to compose cospans.

$$
\begin{array}{lll}
= & \beta;\Pi g;\Pi i & g \text{ is a zigzag,} \\
= & \beta;\Pi(g;i) & \text{functoriality,} \\
= & \beta;\Pi(h;j) & \text{pushout,} \\
= & \beta;\Pi h;\Pi j & \text{functoriality,} \\
= & h;\eta;\Pi j & h \text{ is a zigzag.}
\end{array}
$$

Thus, the outer square formed by Y, U, V and ΠW commutes and couniversality implies the existence of the morphism ρ from W to ΠW. It is a routine calculation that this gives a pushout in the category of coalgebras. This does not require any special properties of Π; it holds in the most general case, that is to say in **Mes**.

The categorical definitions can now be given.

Definition 5.1 *An **event bisimulation** on $\mathcal{S} = (S, \Sigma, h)$ is an epimorphism in the category of coalgebras of Π to some $\mathcal{T}$.*

We have already noted that such arrows are zigzags and that a zigzag induces an event bisimulation on its source $\mathcal{S}$; conversely -and contrary to the span based definition- given an event bisimulation Λ on $\mathcal{S}$, the identity map from $\mathcal{S}$ to (S, Λ, h) will be a zigzag inducing Λ (see Lemmas 3.4, 3.5).

For the case of a binary event bisimulation, that is to say a bisimulation between two different LMPs, we have:

Definition 5.2 *An **event bisimulation** between $\mathcal{S}$ and $\mathcal{S}'$ is a cospan of epimorphism in the category of coalgebras to some object $\mathcal{T}$.*

This can be viewed as ordinary event bisimulation on $\mathcal{S} + \mathcal{S}'$.

Since cospans compose, it is clear that viewed as a relation between LMPs event bisimulation (properly called probabilistic cocongruence) is an equivalence relation. Since our work it has come to our attention that Bartels, Sokolova and de Vink [**?**] have also advocated using cocongruences as the way to define bisimulation. We do not quite have a category of LMPs with cospans as the morphisms since associativity only holds up to isomorphism: we have a bicategory. Incidentally, one can define a nice epi-mono factorization on the category **Mes** that carries over to the category of coalgebras.

Remark 5.3 *It is easy to see that an epimorphim of* **Mes** *that is a morphism of coalgebra is also an epimorphism of coalgebras. The converse is also true. Indeed, let $g : \mathcal{S} \to \mathcal{S}'$ be an epimorphism of coalgebra. We show that its projection into* **Mes** *is a surjection (and hence an epimorphism of* **Mes***). If it is not the case, let $z \in S' \setminus g(S)$ and consider $S'' = (S' \setminus \{z\}) \cup \{z_1, z_2\}$ with the corresponding σ-algebra of sets $\overline{X'}$ for $X' \in \Sigma'$ where $\overline{X'}$ is X' if $z \notin X'$, and $(X' \setminus \{z\}) \cup \{z_1, z_2\}$ otherwise. It remains to define the arrow $h'' : \mathcal{S}'' \to \Pi\mathcal{S}''$. For $s' \in S' \setminus \{z\}$, $h''(s')(\overline{X'}) = h'(s')(X')$. Finally, $h''(z_i)(\overline{X'}) = h'(z)(X')$. Now define $f_i : \mathcal{S}' \to \mathcal{S}''$ as the identity on $S' \setminus \{z\}$ and $f_i(z) = z_i$, $i = 1, 2$. Then $f_1 \circ g = f_2 \circ g$ as coalgebra morphisms and hence $f_1 = f_2$, a contradiction. Hence an epimorphism of coalgebra induces a surjection between the underlying sets of states.*

6 Construction of Mes$''$, Π''

A major virtue of the categorical presentation of the previous section is that one can modify the monad to deal with other, closely related situations. One does not then have to develop the theory again from scratch; one can just use the abstract machinery with a slightly different instantiation. In this section, we modify Giry's monad to deal with negligible sets.

In one of the early papers on LMPs [BDEP97], the question of negligible sets was raised. Consider the LMPs $\mathcal{U} = ([0,1], \mathcal{B}, h)$, where $\mathcal{B}$ are the Borel sets and $h(x, A) = \lambda(A)$ where λ is Lebesgue measure; and $\mathcal{U}'$ with the same state space and σ-algebra but with Markov kernel given by $h'(x, A) = h(x, A)$ if x is irrational and $h'(x, A) = 0$ if x is rational. These two processes behave identically "almost always": except for a set of measure zero they are bisimilar. If we were to observe these systems the probability that we would detect the difference is zero. We would like to formalize this concept by introducing a notion of *almost sure* bisimulation.

There is an obvious gap in the discussion of the last paragraph. According to what measure should one say that the rationals have measure zero? One immediately thinks of Lebesgue measure in this example but what of other examples? Should the state space come equipped with additional structure - perhaps a measure - in order to define what the sets of measure zero are? We introduce such additional structure, but one does not need to introduce a measure just to define the sets of measure zero. Instead we introduce an axiomatically defined class of negligible sets. This can be smoothly incorporated into Giry's monad and then one has the notion of almost sure bisimulation "almost free" with the discussion of the previous section.

6.1 The category $\mathbf{Mes}'$

We introduce in this subsection a refinement, $\mathbf{Mes}'$, of the category $\mathbf{Mes}$ of measurable spaces. The *purpose* of this refinement is to give a means of taking morphisms differing only on negligibly many points to be equal. Refined objects in $\mathbf{Mes}'$ are triples $(S, \Sigma, \mathcal{N})$ where $\mathcal{N}$ is an ideal of negligible sets. An ideal of negligible sets gives rise to a congruence $\sim_{\mathcal{N}}$ on Σ; arrows are required to respect such congruences. When the refined category $\mathbf{Mes}'$ is in place, we will define a quotient category $\mathbf{Mes}''$, where arrows defining the same homomorphism at the level of σ-algebras up to negligibles will be taken to be equal.

This notion of equivalence relaxes the notion introduced earlier [DDLP05]. It is algebraically more natural, as we will see below when we compare the two definitions. Besides, it allows for a nice correspondence with the logical characterization of LMP bisimulation. This will be explained in section 8.

Definition 6.1 *An **object** of* $\mathbf{Mes}'$ *is a triple* $(S, \Sigma, \mathcal{N})$ *where* (S, Σ) *is an object in* $\mathbf{Mes}$*, that is a measurable space, and* $\mathcal{N} \subseteq \Sigma$ *is a* σ*-ideal of* Σ*.*

The additional component in objects, $\mathcal{N}$, is to be thought of intuitively as a set $\mathcal{N}(\nu)$ of *negligible sets* with respect to some measure ν.

We will write $\mathcal{S}$ for an object $(S, \Sigma, \mathcal{N})$, or $(S, \Sigma_S, \mathcal{N}_S)$ and propagate indices or primes as in $\mathcal{S}_1 = (S_1, \Sigma_1, \mathcal{N}_1)$ and $\mathcal{S}' = (S', \Sigma', \mathcal{N}')$.

Suppose we are given such an object $(S, \Sigma, \mathcal{N})$. Clearly the symmetric difference $\triangle$ defines a binary operation over Σ, and it is easily verified that $(\Sigma, \triangle, \cap)$ is a commutative Boolean ring with neutral elements $\varnothing$ and S for $\triangle$ and $\cap$ respectively.

One has the useful following properties of the symmetric difference, where (A_i), (B_i) are families of subsets indexed by the same indexing set I:

$$A^c \triangle B^c \quad = \quad A \triangle B \tag{1}$$

$$\cup_i A_i \triangle \cup_i B_i \quad \subseteq \quad \cup_i (A_i \triangle B_i) \tag{2}$$
$$\cap_i A_i \triangle \cap_i B_i \quad \subseteq \quad \cup_i (A_i \triangle B_i) \tag{3}$$

The first equation is obvious. For the second formula: if x is in the lhs, then without loss of generality, say $x \in A_0$ and $x \notin \cup B_i$, then $x \notin B_0$, so $x \in A_0 \setminus B_0$, so x is in the rhs. For the third formula, without loss of generality, say $x \in \cap A_i$ and $x \notin B_0$, then $x \in A_0 \setminus B_0$, therefore x is in the rhs. Or equivalently, derive it from the first two formulas.

Define the following binary relation over Σ:

$$X \sim_{\mathcal{N}} Y \quad := \quad X \triangle Y \in \mathcal{N}$$

This relation is reflexive (since $\varnothing \in \mathcal{N}$), symmetric (clear) and transitive (because $X \triangle Z \subseteq (X \triangle Y) \cup (Y \triangle Z)$), and thus is an equivalence relation over Σ. Intuitively, two related subsets X and Y are to be thought of as being almost equal. Suppose p is a measure over (S, Σ), then define $\mathcal{N}(p)$ as being the set of $X \in \Sigma$ such that $p(X) = 0$, this is a σ-ideal. Moreover, $d(X, Y) := p(X \triangle Y)$ is a pseudo-metrics on Σ.

A consequence of the properties of the symmetric difference listed above, and the fact that $\mathcal{N}$ is stable under countable unions, is that $\sim_{\mathcal{N}}$ is preserved under Σ's operations, namely complement, countable unions, countable intersections (and also $\triangle$ since it is definable in terms of the preceding operations). We will write $\Sigma/\mathcal{N}$ for the quotient structure. This is no longer a σ-algebra but an object in the category of (abstract) σ-complete Boolean algebra which has as arrows σ-complete morphisms [Mon89, def. 1.28, 5.1]. In this setting, (abstract) σ-ideals can also be seen as kernels for such σ-complete morphisms (referred to below simply as homomorphisms).

Definition 6.2 *An **arrow** f in $\mathbf{Mes}'$ from $(S, \Sigma, \mathcal{N})$ to $(S', \Sigma', \mathcal{N}')$, is a measurable map from (S, Σ) to (S', Σ'), such that one of the three following equivalent conditions hold:*

$$\forall X' \in \mathcal{N}' : f^{-1}(X') \in \mathcal{N} \tag{4}$$
$$\forall X', Y' \in \Sigma' : X' \sim_{\mathcal{N}'} Y' \Rightarrow f^{-1}X' \sim_{\mathcal{N}} f^{-1}Y' \tag{5}$$
$$f^{-1} \text{ is an homomorphism from } \Sigma'/\mathcal{N}' \to \Sigma/\mathcal{N} \tag{6}$$

We have to prove that these three conditions are equivalent.
(5) $\Rightarrow$ (4) : pick $X' \in \mathcal{N}'$, then $X' \sim_{\mathcal{N}'} \varnothing$, so $f^{-1}X' \sim_{\mathcal{N}} \varnothing$, hence $f^{-1}X' \in \mathcal{N}$.
(5) $\Leftarrow$ (4) : pick X', $Y' \in \Sigma'$ such that $X' \sim_{\mathcal{N}'} Y'$, then $X' \triangle Y' \in \mathcal{N}'$, so $f^{-1}(X' \triangle Y') = f^{-1}X' \triangle f^{-1}Y' \in \mathcal{N}$, so $f^{-1}X' \sim_{\mathcal{N}} f^{-1}Y'$.
(6) $\Leftrightarrow$ (5) : clear.

The additional condition (4) is obviously stable by composition, so our data actually defines a category.

Condition (4) results in a serious restriction on **Mes** arrows. For instance, a measurable map f from $(S, \Sigma, \mathcal{N})$ to $([0,1], \mathcal{B}, \mathcal{N}(\lambda))$, where λ is the Lebesgue measure, and with countable image, will violate this condition unless S itself is in $\mathcal{N}$, because countable subsets of $[0,1]$ are all in $\mathcal{N}(\lambda)$. In particular, no measurable endomap on $([0,1], \mathcal{B}, \mathcal{N}(\lambda))$ with countable image is in $\mathbf{Mes}'$.

On the other hand, if $\mathcal{N}_{\mathcal{S}} = \{\varnothing\}$, then $\sim_{\mathcal{N}_{\mathcal{S}}}$ is the identity relation, and any measurable map to $\mathcal{S}$ will verify (5).

Supposing in the definition above, both $\mathcal{N}$ and $\mathcal{N}'$ are given by measures p and q, we can rephrase the condition on arrows by saying that $f(p) \ll q$, where $f(p)$ is the image measure under f. Indeed, (4) reads now as $q(N) = 0$ implies $p(f^{-1}(N)) = 0$.

We now have our refined category $\mathbf{Mes}'$, and the next thing is to check whether Giry's monad can be extended to this new setting.

6.2 Refining Giry's monad

For $(S, \Sigma, \mathcal{N})$ in $\mathbf{Mes}'$, $\mathbf{\Pi}'S$ and $\Sigma_{\mathbf{\Pi}'S}$ are given as in $\mathbf{Mes}$; one chooses the additional component $\mathcal{N}_{\mathbf{\Pi}'S}$ to be generated by the following particular subsets N_A of measures:

$$N_A \quad := \quad \{\nu \mid \nu(A) > 0\}. \tag{7}$$

where $A \in \mathcal{N}$. Since $A \in \mathcal{N} \subseteq \Sigma$, and $N_A = e_A^{-1}(0, 1]$, one has $N_A \in \Sigma_{\mathbf{\Pi}'S}$, so our collection of negligible sets is adapted to the σ-algebra structure. The intent of this definition is to neglect the measures that ascribe nonzero weight to the negligible sets.

Equivalently one can define $\mathcal{N}_{\mathbf{\Pi}'S}$ as $\mathcal{N}\{e_A^{-1}(\alpha, 1];\, \alpha \in [0, 1],\, A \in \mathcal{N}\}$, which nicely parallels the definition of $\Sigma_{\mathbf{\Pi}S}$ as $\sigma\{e_A^{-1}(\alpha, 1];\, \alpha \in [0, 1],\, A \in \Sigma\}$.

Had we chosen to define negligible sets via a measure, we would need one here. When is an $\mathcal{N}$ obtainable as an $\mathcal{N}(\nu)$ for some ν is not so clear. In any case, such a ν would only intervene in the theory through sets of ν measure 0, so any two equivalent (meaning mutually absolutely continuous) measures would define isomorphic objects.

The definition of $\Pi'(f)$ is the same as that of Π, so functoriality of Π is obvious. We also have to verify that Π and its associated natural transformations satisfy (4), which follows from:

Lemma 6.3 *For all $A \in \Sigma$:*
— *(i) $\Pi(f)^{-1}(N_A) = N_{f^{-1}(A)}$*
— *(ii) $\eta^{-1}(N_A) = A$;*
— *(iii) $\mu^{-1}(N_A) = N_{N_A}$.*

Proof . (i) $\nu \in \Pi(f)^{-1}(N_A) \Leftrightarrow \Pi(f)(\nu) \in N_A \Leftrightarrow \Pi(f)(\nu)(A) > 0 \Leftrightarrow \nu(f^{-1}(A)) > 0 \Leftrightarrow \nu \in N_{f^{-1}(A)}$.
(ii) $a \in \eta^{-1}(N_A) \Leftrightarrow \delta_a(A) > 0 \Leftrightarrow a \in A$.
(iii) $P \in \mu^{-1}(N_A) \Leftrightarrow \mu(P)(A) > 0 \Leftrightarrow \int e_A dP > 0 \Leftrightarrow \int \mathbf{1}_{N_A} dP > 0$ since $N_A := \{e_A > 0\}$, and the last statement says precisely $P(N_A) > 0$, that is to say $P \in N_{N_A}$. ∎

Naturality requirements and other conditions are still valid, so $\mathbf{\Pi}'$ is a monad refining $\mathbf{\Pi}$.

6.3 The quotient category Mes″

One now defines the quotient category $\mathbf{Mes}''$: objects are as in $\mathbf{Mes}'$; arrows are $\sim$ equivalence classes where, for $f,\, g : (S, \Sigma, \mathcal{N}) \to (S', \Sigma', \mathcal{N}')$:

$$f \sim g \quad := \quad \forall X' \in \Sigma' \;\; f^{-1}(X') \sim_{\mathcal{N}} g^{-1}(X'). \tag{8}$$

In other words arrows that correspond to the same homomorphism from $\Sigma'/\mathcal{N}' \to \Sigma/\mathcal{N}$ are identified. This definition of $f \sim g$ differs from our first attempt [DDLP05], where f and g were taken to be equivalent only when $\{f \neq g\}$ is included in a negligible of $\mathcal{N}$. This is in fact too strong since, as we will see later, it does not lead to a reasonable notion of epimorphism. In fact, it is equivalent to requiring that

$$\exists N \in \mathcal{N} : \textstyle\bigcup_{x \in S} f^{-1}(\{x\}) \bigtriangleup g^{-1}(\{x\}) \subset N,$$

which involves a very big union that was required to be negligible. However, the two notions coincide when the target object has a countably generated σ-algebra which separates points, in particular for analytic spaces. For example, when $S' = [0,1]$ with the Borel σ-algebra, we have that $\{f \neq g\} = \cup_r f^{-1}[r,1] \triangle g^{-1}[r,1]$. This will be helpful when we will work with kernels.[3]

Example 6.4 *Let $\mathcal{S} = (S, \Sigma_S, \mathcal{N}_S)$ be an object in $\mathbf{Mes}''$, and let X, Y be subsets of S. Define $\mathbf{1}_X : \mathcal{S} \to (\{0,1\}, \wp(\{0,1\}), \varnothing)$, the indicator function of X, as $\mathbf{1}_X(x) = 1$ if $x \in X$, and 0 else. Then $\mathbf{1}_X$ is an arrow of $\mathbf{Mes}''$ iff $X \in \Sigma_S$, and $\mathbf{1}_X \sim \mathbf{1}_Y$ iff $X \triangle Y \in \mathcal{N}_S$.*

Example 6.5 *η the unit of Π is equivalent to $0 : S \to \Pi(S)$, with $0(s)$ being the null subprobability iff $\Sigma_S = \mathcal{N}_S$.*

6.4 Compatibility of composition with the quotient

We first have to verify that the equivalence that we just defined is compatible with composition.

Lemma 6.6 *The equivalence $\sim$ defined in (8) is stable by composition.*

Proof . Let $f_1 : (S_1, \Sigma_1, \mathcal{N}_1) \to (S, \Sigma, \mathcal{N})$, $g, h : (S, \Sigma, \mathcal{N}) \to (S', \Sigma', \mathcal{N}')$ and $f_2 : (S', \Sigma', \mathcal{N}') \to (S_2, \Sigma_2, \mathcal{N}_2)$.

$g \sim h \Rightarrow gf_1 \sim hf_1$: pick $X' \in \Sigma'$, one has

$$f_1^{-1} g^{-1}(X') \triangle f_1^{-1} h^{-1}(X') = f_1^{-1}\Big(g^{-1}(X') \triangle h^{-1}(X')\Big)$$

and since $g^{-1}(X') \triangle h^{-1}(X') \in \mathcal{N}$, one concludes by condition (4).
$g \sim h \Rightarrow f_2 g \sim f_2 h$: a straightforward consequence of the measurability of f_2. ∎

6.5 Π preserves ∼

The next thing to check is whether our probabilistic functor also preserves the equivalence between maps. In order to do this, it will be convenient to first prove a set-theoretic lemma.

Lemma 6.7 *Let S be a set, and $\mathcal{N}$ be a set of subsets of S closed under countable unions and subset; let T be a set, and f, g be maps from S to T. Define Σ_T as the set of subsets X of T such that $f^{-1}X \triangle g^{-1}X \in \mathcal{N}$, then Σ_T is a σ-algebra.*

Proof . Obviously $T \in \Sigma_T$, since $f^{-1}(T) \triangle g^{-1}(T) = \varnothing$. To prove what we want, it is enough to prove that Σ_T is closed under complement and countable unions.

Suppose $X \in \Sigma_T$, then using equation (1), we get:

$$f^{-1}(X^c) \triangle g^{-1}(X^c) = f^{-1}(X)^c \triangle g^{-1}(X)^c = f^{-1}(X) \triangle g^{-1}(X)$$

so X^c is in Σ_T as well.

[3] Actually, one could restrict the present category to that of ω-generated σ-algebras; it is readily seen that these are preserved under Π. In this subcategory, the two notions of equivalence are the same.

Suppose X_i is a countable family in Σ_T, then using inequation (2), we get:

$$f^{-1}(\cup_i X_i) \triangle g^{-1}(\cup_i X_i) = \bigcup_i f^{-1}(X_i) \triangle \bigcup_i g^{-1}(X_i) \subseteq \bigcup_i (f^{-1}(X_i) \triangle g^{-1}(X_i))$$

Since for all i, $f^{-1}(X_i) \triangle g^{-1}(X_i) \in \mathcal{N}$, and $\mathcal{N}$ is stable under countable unions and subset, one sees that $\cup_i X_i$ is in Σ_T as well. ∎

Now we turn to the proof that $\mathbf{\Pi}$ respects $\sim$.

Lemma 6.8 *Let f, g be arrows from $\mathcal{S} = (S, \Sigma_S, \mathcal{N}_S)$ to $\mathcal{T} = (T, \Sigma_T, \mathcal{N}_T)$, then: $f \sim g \Rightarrow \mathbf{\Pi}(f) \sim \mathbf{\Pi}(g)$.*

Proof . We have to show that $\mathbf{\Pi}(f)^{-1}(W) \triangle \mathbf{\Pi}(g)^{-1}(W) \in \mathcal{N}_{\mathbf{\Pi}(S)}$ for all $W \in \Sigma_{\mathbf{\Pi}(T)}$. Define:

$$W_{A,r} := e_A^{-1}(r, 1] = \{\nu \mid \nu \in \mathbf{\Pi}(T) \text{ and } \nu(A) > r\}$$

for $A \in \Sigma_T$ and $r \in \mathbb{Q} \cap [0, 1]$. By definition, $\Sigma_{\mathbf{\Pi}(T)}$ is the smallest σ-algebra containing the $W_{A,r}$, and therefore by the preceding lemma, it suffices to show what we want to show for these particular generating sets.

One has $\nu \in \mathbf{\Pi}(f)^{-1}(W_{A,r})$ iff $\nu(f^{-1}(A)) > r$, therefore:

$$\begin{array}{ll}
\nu \in \mathbf{\Pi}(f)^{-1}(W_{A,r}) \triangle \mathbf{\Pi}(g)^{-1}(W_{A,r}) & \Leftrightarrow \\
(\nu(f^{-1}(A)) > r \wedge \nu(g^{-1}(A)) \leq r) \vee (\nu(f^{-1}(A)) \leq r \wedge \nu(g^{-1}(A)) > r) & \Rightarrow \\
\nu(f^{-1}(A)) \neq \nu(g^{-1}(A)) & \Rightarrow \\
\nu(f^{-1}(A) \triangle g^{-1}(A)) > 0 &
\end{array}$$

To see why the last step is correct, one argues as follows: by additivity of ν, $\nu(X \cup Y) = \nu(X \triangle Y) + \nu(X \cap Y)$, so if $\nu(X \triangle Y) = 0$, one has $\nu(X \cup Y) = \nu(X \cap Y)$, and then $\nu(X) = \nu(Y)$, since $\nu(X \cap Y) \leq \nu(X) \leq \nu(X \cup Y)$.

It now follows from $f \sim g$ and $A \in \Sigma_T$, that $f^{-1}(A) \triangle g^{-1}(A) \in \mathcal{N}$. So $\nu(f^{-1}(A) \triangle g^{-1}(A)) > 0$, and $\nu \in N_{f^{-1}(A) \triangle g^{-1}(A)} \in \mathcal{N}_{\mathbf{\Pi}(S)}$; $\mathcal{N}_{\mathbf{\Pi}(S)}$ being downward closed in $\Sigma_{\mathbf{\Pi}(S)}$, we are done. ∎

Since we are working here with a notion of equivalence for morphisms which is larger than the one given in our previous paper [DDLP05], it makes the statement above stronger and a bigger machinery is needed in the proof. However, as said before, the point of this more flexible definition is not to have a more complicated argument, but to derive a more satisfying notion of epimorphism.

6.6 Epimorphisms

The following proposition says that the only reason why $f^{-1}X$ may be small when f is an epimorphism is that X is small itself, which means X only grows under f^{-1}, which in turn is fairly intuitive as a description of a sort of "fuzzy surjection." Note also that this condition is converse to condition (4).

Proposition 6.9 *Let $f : (S, \Sigma, \mathcal{N}) \to (S', \Sigma', \mathcal{N}')$ be an arrow of* $\mathbf{Mes}''$*, the following are equivalent:*

$$f \textit{ is an epimorphism} \tag{9}$$

$$f^{-1} \textit{ is a monomorphism from } \Sigma'/\mathcal{N}' \to \Sigma/\mathcal{N} \tag{10}$$

$$\forall X' \in \Sigma' : f^{-1}(X') \in \mathcal{N} \Rightarrow X' \in \mathcal{N}' \tag{11}$$

$$\forall X', Y' \in \Sigma' : f^{-1}X' \sim_{\mathcal{N}} f^{-1}Y' \Rightarrow X' \sim_{\mathcal{N}'} Y' \tag{12}$$

Proof . (9) $\Rightarrow$ (11): Let f be an epimorphism and $X \in \Sigma'$ such that $f^{-1}(X) \in \mathcal{N}$. Now consider the two morphisms 1_X and $1_\varnothing : (S', \Sigma', \mathcal{N}') \rightarrow (\{0,1\}, \mathcal{P}\{0,1\}, \{\varnothing, \{1\}\})$. We have that $1_X f \sim 1_\varnothing f$ because $(1_X f)^{-1}(\{1\}) \triangle (1_\varnothing f)^{-1}(\{1\}) = f^{-1}(X) \triangle \varnothing = f^{-1}(X) \in \mathcal{N}$, and since f is an epimorphism, $1_X \sim 1_\varnothing$, so $(1_X)^{-1}(\{1\}) \triangle (1_\varnothing)^{-1}(\{1\}) = X \triangle \varnothing = X \in \mathcal{N}'$.
(9) $\Leftarrow$ (11): Let $g, h : (S', \Sigma', \mathcal{N}') \rightarrow (S'', \Sigma'', \mathcal{N}'')$ be such that $gf \sim hf$. For any $X \in \Sigma''$ we have

$$f^{-1}(g^{-1}(X) \triangle h^{-1}(X)) = (gf)^{-1}(X) \triangle (hf)^{-1}(X) \in \mathcal{N}$$

and condition (11) implies that $g^{-1}(X) \triangle h^{-1}(X) \in \mathcal{N}'$, so $g \sim h$.
It is clear that (10), (11) and (12) are equivalent. ∎

Note that, as discussed after Equation (8), with a more stringent notion of $\sim$ an epimorphism would satisfy the implication in Equation (11) for all $X' \subseteq S'$, instead of only for measurable sets. This would be a very strong condition for a morphism to be an epimorphism. Indeed, when we will take the quotient with respect to our logic, the quotient morphism will be an epimorphism according to the new definition, but not with respect to the stronger one.

7 $\mathbf{\Pi}''$ coalgebras and bisimulation

Bisimulations give means to identify points in a given LMP, while negligibles give means to identify measurable sets; the object of the present section is to show how the latter can be combined with the former.

7.1 LMPs with negligibles

When we worked in **Mes**, a labelled process was a $\mathbf{\Pi}$-coalgebra. This coalgebraic presentation was developed by de Vink and Rutten [dVR97, **?**] and noted in passing in [BDEP97]. We now have the following natural extension.

Definition 7.1 *A* ***labelled Markov process with negligibles (LMPn)*** *is a* $\mathbf{\Pi}''$*-coalgebra.*

We recapitulate what this means.

Lemma 7.2 *Equivalently, an LMPn is a structure* $(S, \Sigma, \mathcal{N}, h)$*, where* S *is the set of states,* Σ *is a* σ*-algebra on* S*, and* $h : S \times \Sigma \longrightarrow [0,1]$ *satisfies*

1. *for all* $s \in S$*,* $h(s) : \Sigma \rightarrow [0,1]$ *is a subprobability measure;*
2. *for all* $A \in \Sigma$*,* $h(\cdot)(A) : S \rightarrow [0,1]$ *is measurable;*
3. *for all* $A \in \mathcal{N}$*,* $\{s \mid h(s)(A) > 0\} \in \mathcal{N}$*.*

Proof . 1: Clear from the definition of $\mathbf{\Pi}S$.
2: A morphism $h : (S, \Sigma, \mathcal{N}) \rightarrow \mathbf{\Pi}''(S, \Sigma, \mathcal{N}))$ must satisfy that for all measurable W of $\mathbf{\Pi}S$, $h^{-1}W \in \Sigma$. In particular, for $W = \{\nu \mid e_A(\nu) > r\} = \{\nu \mid \nu(A) > r\}$. But $h^{-1}W = \{s \mid h(s)(A) > r\}$. So this is just the condition that $h(\cdot, A)$ be a measurable function from S to $[0,1]$.
3: By definition for all $A \in \mathcal{N}$, $h(\cdot, A)^{-1}(N_A) \in \mathcal{N}$, which means that $\{s \mid h(s, A) > 0\} \in \mathcal{N}$. ∎

One will notice that Condition 2 is equivalent to the stability of Σ and Condition 3 is equivalent to the stability of $\mathcal{N}$. Indeed, Condition 3 implies $\{s \mid h(s)(A) > r\} \in \mathcal{N}$ for all $r \in \mathbb{Q} \cap [0,1]$ and $A \in \mathcal{N}$, because of Condition 2 and by the fact that $\mathcal{N}$ is a σ-ideal.

When negligibles are given via some measure ν defined on (S, Σ), that is to say $\mathcal{N} = \mathcal{N}_\nu$, then Condition 3 amounts to

$$\nu(A) = 0 \quad \Rightarrow \quad \nu(\{s \mid h(s, A) > 0\}) = 0,$$

which says that "negligibly many points may jump to a negligible set".

Lemma 7.3 *Suppose h is a kernel, and ν dominates h in the sense that for all s, $h(s, \cdot) \ll \nu$, then $\mathcal{N}_\nu$ is stable under h, that is to say:*

$$\forall B \in \mathcal{N}_\nu : \langle a \rangle_0(B) \in \mathcal{N}_\nu.$$

Proof . If $\nu(B) = 0$, then for all s, $h(s, B) = 0$, so that $\langle a \rangle_0(B) = \varnothing \in \mathcal{N}_\nu$. ∎

So the condition of stability with respect to negligibles is an abstract form of domination.

Recall that one interprets $h(s)(X)$ (often written $h(s, X)$) as the probability of the process starting in state s making a transition into one of the states in X.

Lemma 7.4 *A morphism $f : (S, \Sigma, \mathcal{N}, h) \to (S', \Sigma', \mathcal{N}', h')$ is a morphism of coalgebras iff for all $A' \in \Sigma'$*

$$\{h(\cdot, f^{-1}A') \neq h'(f(\cdot), A')\} \in \mathcal{N}$$

Proof . For all $A' \in \Sigma'$, and all B Borel set of $[0, 1]$:

$$\begin{aligned}
& ((\mathbf{\Pi}'' f)h)^{-1}\{\nu \mid \nu(A') \in B\} \sim (h'f)^{-1}\{\nu \mid \nu(A') \in B\} \\
\Leftrightarrow \quad & h^{-1}\{\mu \mid (\mathbf{\Pi}'' f)\mu(A') \in B\} \sim f^{-1}\{s' \mid h'(s', A') \in B\} \\
\Leftrightarrow \quad & h^{-1}\{\mu \mid \mu(f^{-1}A') \in B\} \sim f^{-1}\{s' \mid h'(s', A') \in B\} \\
\Leftrightarrow \quad & \{s \mid h(s, f^{-1}A') \in B\} \sim \{s \mid h'(fs, A') \in B\}
\end{aligned}$$

So that $(\mathbf{\Pi}'' f)h \sim h'f$ iff $h(\cdot, f^{-1}A') \sim h'(f(\cdot), A')$, which is iff $\{h(\cdot, f^{-1}A') \neq h'(f(\cdot), A')\} \in \mathcal{N}$, because $([0, 1], \mathcal{B})$ is ω-generated and separates points. ∎

Recall (from def. 4.2) that for a given kernel h, and leaving aside the labels, $\langle\rangle_r$ is defined as $\langle\rangle_r(A) := \{s \mid h(s, A) > r\}$; these endomaps of Σ_S (which are not Boolean algebra homomorphisms) are compatible with the quotient $\Sigma/\mathcal{N}$.

Lemma 7.5 *Let $(S, \Sigma, \mathcal{N}, h)$ be an LMPn, A, B in Σ, if $A \sim_\mathcal{N} B$ then $\langle\rangle_r(A) \sim_\mathcal{N} \langle\rangle_r(B)$.*

Proof . Define $\Box_r : \Sigma_S \to \Sigma_{\Pi(S)}$ as $\Box_r(A) := \{p \mid p(A) > r\}$, this $[0, 1]$-indexed family of maps preserves $\sim$, because if $A \sim_\mathcal{N} B$, then:

$$\{p \mid p(A) > r \wedge p(B) \leq r\} \subseteq \{p \mid p(A \setminus B) > 0\} \subseteq N_{A \setminus B}$$

so $\Box_r(A) \bigtriangleup \Box_r(B) \subseteq N_{A\setminus B} \cup N_{B \setminus A} \in \mathcal{N}_{\Pi(S)}$. Now $\langle\rangle_r = h^{-1} \circ \Box_r : \Sigma_S \to \Sigma_S$ must also preserve $\sim$, since by definition of an arrow h^{-1} does. ∎

Here is a another argument: suppose $A = B \triangle N$ for some $N \in \mathcal{N}$, then $\langle\rangle_r(B \triangle N) \triangle \langle\rangle_r(B) \subseteq \langle\rangle_0(N)$ and $\langle\rangle_0(N) \in \mathcal{N}$ (by Lemma 7.2).

Before defining bisimulation, we introduce a negligible version of stability. Working with negligible sets, it is natural that we are led to work with $\Lambda + \mathcal{N}$.

Definition 7.6 *Given an LMPn $(S, \Sigma, \mathcal{N}, h)$ and $\Lambda \subseteq \Sigma$ a σ-algebra, one says that Λ is* ***almost stable*** *if $\Lambda + \mathcal{N}$ is stable, or equivalently if for all $A \in \Lambda$, $r \in [0,1]$, $\langle\rangle_r(A) \in \Lambda + \mathcal{N}$.*

Clearly saying $\Lambda + \mathcal{N}$ is stable implies the second condition; conversely, if $A \triangle N \in \Lambda + \mathcal{N}$, then by the previous lemma, $\langle\rangle_r(A \triangle N) \sim_{\mathcal{N}} \langle\rangle_r(A)$ which is in $\Lambda + \mathcal{N}$ by the second condition.

7.2 Bisimulation and zizag morphisms

Event bisimulation is defined as before.

Definition 7.7 *An* ***event bisimulation*** *on an LMPn $(S, \Sigma, \mathcal{N}, h)$ is a sub-σ-algebra Λ of Σ such that $(S, \Lambda + \mathcal{N}, \mathcal{N}, h)$ is an LMPn, or equivalently such that $\Lambda + \mathcal{N}$ is stable.*

The equivalence is clear.[4] Here, one could define event bisimulation by asking $(S, \Lambda, \Lambda \cap \mathcal{N}, h)$ to be an LMP, or equivalently that Λ is stable, but the proofs seem more difficult. Yet, it follows from Noether's second isomorphism [**?**] that $(\Lambda + \mathcal{N})/\mathcal{N}$ and $\Lambda/(\Lambda \cap \mathcal{N})$ are isomorphic as σ-complete Boolean algebras, so both approaches could amount to the same.

For the case of a binary event bisimulation (*i.e.*, between two different LMPn) we follow the categorical lead by using cospans of zigzags. This will make bisimulation an equivalence relation, as we will explain below.

Definition 7.8 *A* ***zigzag morphism*** *is a coalgebra epimorphism; an* ***event bisimulation between $\mathcal{S}$ and $\mathcal{S}'$*** *is a cospan of zigzag morphisms to some $\mathcal{T}$.*

Remark 7.9 *As for Remark 5.3 we argue that a coalgebra epimorphism is exactly a coalgebra morphism that is an epimorphism in* $\mathbf{Mes}''$*. That the second implies the former is clear. Now let $g : \mathcal{S} \to \mathcal{S}'$ be an epimorphism of coalgebra. We show that its projection into* $\mathbf{Mes}''$ *is an epimorphism in* $\mathbf{Mes}''$*. Let $X' \in \Sigma'$ such that $g^{-1}(X') \in \mathcal{N}$ and consider $S'' = (S' \backslash X') \cup (X' \times \{1,2\})$ with the corresponding σ-algebra of sets $\overline{Y'}$ with $Y' \in \Sigma'$ where $\overline{Y'} = Y' \setminus X' \cup ((Y \cap X') \times \{1,2\})$. It remains to define the arrow $h'' : \mathcal{S}'' \to \Pi''\mathcal{S}''$. For $s' \in S' \setminus X'$, $h''(s')(\overline{Y'}) := h'(s')(Y')$. Finally, $h''((z,i))(\overline{X'}) = h'(z)(X')$. Now define $f_i : \mathcal{S}' \to \mathcal{S}''$ as the identity on $S' \setminus X'$ and for $z \in X'$, $f_i(z) = (z,i)$, $i = 1,2$. Then $f_1 \circ g \sim f_2 \circ g$ as coalgebra morphisms and hence $f_1 \sim f_2$, implying that $X' = f_1^{-1}(\overline{X'}) \triangle f_2^{-1}(\overline{X'}) \in \mathcal{N}'$, as wanted.*

The relation between the unary and binary versions of bisimulations is given by the canonical correspondence between zigzag morphisms and unary event bisimulations (right below), together with the fact that a binary event bisimulation can also be seen as a zigzag from the disjoint sum $\mathcal{S} + \mathcal{S}'$ to $\mathcal{T}$.

Proposition 7.10 *Any event bisimulation on $\mathcal{S} = (S, \Sigma_S, \mathcal{N}_S, h_S)$ is given as $f^{-1}(\Sigma_T)$ with f a zigzag morphism to some $\mathcal{T}$, and conversely any coalgebra morphism f from $\mathcal{S}$ to some $\mathcal{T}$ defines an event bisimulation given as $f^{-1}(\Sigma_T)$.*

[4] Conditions 1, 2 are clear, and 3 also since $\langle\rangle_r(N \in \mathcal{N}) \in \mathcal{N}$ because it is true for $(S, \Sigma, \mathcal{N}, h)$.

Proof . For the first point, suppose Λ is an event bismulation, the identity from $\mathcal{S}$ to $\mathcal{T} = (S, \Lambda + \mathcal{N}_S, \mathcal{N}_S, h_S)$ is clearly a morphism since $\Lambda + \mathcal{N}_S \subseteq \Sigma_S$, and clearly also an epimorphism. Conversely, suppose $f : \mathcal{S} \to \mathcal{T}$ is a coalgebra morphism, we want to prove that $f^{-1}(\Sigma_T) \subseteq \Sigma_S$ is an event bisimulation on $\mathcal{S}$, which means $\langle\rangle_r(f^{-1}(A')) \in f^{-1}(\Sigma_T) + \mathcal{N}_S$ for every $A' \in \Sigma_T$. Pick A' in Σ_T, since f is a coalgebra morphism:

$$\begin{array}{rcl} \langle\rangle_r(f^{-1}(A')) & := & \{s \mid h_S(s, f^{-1}(A')) > r\} \\ & \sim_{\mathcal{N}_S} & \{s \mid h_T(f(s), A') > r\} \\ & = & f^{-1}\{s' \mid h_T(s', A') > r\} \in f^{-1}(\Sigma_T) \end{array}$$

as needed. ∎

Theorem 7.11 *Bisimilarity is an equivalence relation.*

The proof is - just as before - given by the fact that if we work with cospans then we can compose using pushouts and this does not require Π'' to preserve anything. Everything works out in this case by the same abstract arguments as before. We argued earlier that cospans are the natural structure to use if one is interested in equivalence relations.

Since cospans compose, it is clear that viewed as a relation between LMPns event bisimulation is an equivalence relation just as with LMPs. Once again we do not quite have a category of LMPns with cospans as the morphisms since associativity only holds up to isomorphism: we have a bicategory.

8 Logical characterization revisited

Recall that the logic $\mathcal{L}$ has the following syntax:

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a\rangle_r \phi$$

where a is an action and r is a rational between 0 and 1.

Proposition 4.5 remains true for LMPn, and we can further say that $\sigma([\![\mathcal{L}]\!]) + \mathcal{N}$ is the smallest stable σ-algebra containing $\mathcal{N}$ and included in Σ, that is to say, for every stable $\Lambda \subseteq \Sigma$, if $\Lambda \supseteq \mathcal{N}$, then $\Lambda \supseteq \sigma([\![\mathcal{L}]\!]) + \mathcal{N}$ (since Λ is stable, $\Lambda \supseteq \sigma([\![\mathcal{L}]\!])$, and since it contains also $\mathcal{N}$, it must contain the smallest σ-algebra containing $\sigma([\![\mathcal{L}]\!])$ and $\mathcal{N}$ which we know is $\sigma([\![\mathcal{L}]\!]) + \mathcal{N}$); and this entails, that for every *almost* stable $\Lambda \subseteq \Sigma$, one has $\Lambda + \mathcal{N} \supseteq \sigma([\![\mathcal{L}]\!]) + \mathcal{N}$. Note however, that we don't know whether $\sigma([\![\mathcal{L}]\!]) + \mathcal{N}$ is the smallest almost stable σ-algebra.

The notion of logical equivalence between states was trivial to define, but it is not as easy when turning to the logical equivalence of two LMPns. For LMPs, one says that two LMPs are logically equivalent if every state of any of them is equivalent to some state of the other ($s_1 \in S_1$ and $s_2 \in S_2$ being equivalent, written $s_1 \approx s_2$, if they satisfy exactly the same formulas of $\mathcal{L}$). Now that we work with negligibles, bisimilarity is less demanding (since it is easier to be a zigzag because of the equivalence between arrows in $\mathbf{Mes}'$), therefore, to find the matching notion on the logical side, we have to relax the usual notion of logical equivalence. This will be based on the following lemma about "traditional" LMPs.

Lemma 8.1 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be LMPs, and define the relation ρ over $[\![\mathcal{L}]\!]_{\mathcal{S}_1} \times [\![\mathcal{L}]\!]_{\mathcal{S}_2}$ as $\rho := \{([\![\theta]\!]_{\mathcal{S}_1}, [\![\theta]\!]_{\mathcal{S}_2}) \mid \theta \in \mathcal{L}\}$, then $\mathcal{S}_1 \approx_{\mathcal{L}} \mathcal{S}_2$ iff ρ extends to a σ-complete isomorphism between $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_1})$ and $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_2})$.*

Proof . Suppose first $\mathcal{S}_1 \approx_{\mathcal{L}} \mathcal{S}_2$. Consider a test $\tau = \wedge\theta_i^{\epsilon_i}$, where the conjunction ranges over $\mathcal{L}$ and we use the notation $\epsilon_i \in \{\pm 1\}$ and $[\![\theta_i^{-1}]\!] := ([\![\theta]\!]_i)^c$. By definition of $\approx_{\mathcal{L}}$, it is inhabited in $\mathcal{S}_1$ iff it is in $\mathcal{S}_2$, and if inhabited it is an atom of both $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_1})$ and $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_2})$. To see why tests with non empty denotation are atoms, observe first that two tests cannot denote equal sets unless they are empty, and second that $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_1})$ is by definition generated by the $[\![\theta]\!]_{\mathcal{S}_1}$s which are closed under the equivalence relation generated by the non-empty tests (because these are possibly uncountable unions of tests). Hence, the relation $\rho' = \{[\![\tau]\!]_1, [\![\tau]\!]_2\}$ restricts to a bijection between atoms of the respective algebras. Pick now a formula θ: θ can be written as a union of tests, those that set θ to true; therefore θ and θ' denote the same subset of S_1 iff the tests that set one to true and the other to false are empty, which is iff they denote also the same subset of S_2, because tests are equally inhabited in both LMPs. We infer that ρ is a bijection. Now both algebras have the same atoms up to ρ', and their generators, given by the θ with non empty denotations write as the same unions of atoms, so there must be an isomorphism between them extending both ρ and ρ'.

Suppose now there exists an isomorphism $\hat{\rho}$ extending ρ. Pick a test $\tau = \wedge\theta_i^{\epsilon_i}$ inhabited in $\mathcal{S}_1$, it maps to an atom of $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_2})$ under $\hat{\rho}$ given by $\cap[\![\theta_i]\!]_{\mathcal{S}_2}^{\epsilon_i}$ because $\hat{\rho}$ is a σ-complete homomorphism, so this atom is the denotation of τ in $\mathcal{S}_2$. ∎

This reformulation gives the key to the definition of logical equivalence for LMPns. We use the following notation

$$\sigma([\![\mathcal{L}]\!]_{\mathcal{S}})/\mathcal{N} := \frac{\sigma([\![\mathcal{L}]\!]_{\mathcal{S}}) + \mathcal{N}}{\mathcal{N}}$$

for the set of $\sim_{\mathcal{N}}$-equivalence classes of $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}}) + \mathcal{N}$ (of course, it may not be the case that $\mathcal{N} \subseteq \sigma([\![\mathcal{L}]\!]_{\mathcal{S}})$). Just as for LMPs, formulas define a canonical relation ρ between $\sigma([\![\mathcal{L}]\!]_1)/\mathcal{N}_1$ and $\sigma([\![\mathcal{L}]\!]_2)/\mathcal{N}_2$ given as $\rho := \{([[\![\theta]\!]_1]_{\mathcal{N}_1}, [[\![\theta]\!]_2]_{\mathcal{N}_2}) \mid \theta \in \mathcal{L}\}$, relating the respective equivalence classes of $[\![\theta]\!]_1$ and $[\![\theta]\!]_2$.

Definition 8.2 *Let $\mathcal{S}_1$, and $\mathcal{S}_2$ be LMPns, one says $\mathcal{S}_1 \approx_{\mathcal{L}} \mathcal{S}_2$ if ρ extends to a σ-complete isomorphism between $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_1})/\mathcal{N}_1$ and $\sigma([\![\mathcal{L}]\!]_{\mathcal{S}_2})/\mathcal{N}_2$.*

By Lemma 8.1, if only the empty set is taken as negligible, one gets back the ordinary notion of logical equivalence for LMPs. Even if this definition looks more abstract than the one for ordinary LMPs, it can be formulated in the same way for the particular case of discrete countable systems. Indeed, since in this case negligible sets are exactly unions of negligible singletons and since the isomorphism involved in the definition is σ-complete, Definition 8.2 is equivalent to requiring that every *non-negligible* state of any of them is logically equivalent to some *non-negligible* state of the other.

We now set up to prove the logical characterization of binary event bisimulation for LMPns. We will first prove that if there is a cospan of zigzag morphisms between two LMPns, then these are logically equivalent with respect to Definition 8.2. To do this, we only have to show that it is the case for one zigzag morphism. We first need the following lemma.

Lemma 8.3 *If $f : \mathcal{S} \to \mathcal{S}'$ is a morphism of coalgebras, then for all $\theta \in \mathcal{L}$, we have*

$$f^{-1}([\![\theta]\!]_{\mathcal{S}'}) \triangle [\![\theta]\!]_{\mathcal{S}} \in \mathcal{N}.$$

Proof . The proof is by induction on θ. For $\top$: $[\![\top]\!] \triangle f^{-1}[\![\top]\!]_{S'} = \varnothing$. For $\theta_0 \wedge \theta_1$: by inequation (3), $([\![\theta_0]\!] \cap [\![\theta_1]\!]) \triangle (f^{-1}[\![\theta_0]\!]_{S'} \cap f^{-1}[\![\theta_1]\!]_{S'})$ is included in $([\![\theta_0]\!] \triangle f^{-1}[\![\theta_0]\!]_{S'}) \cup ([\![\theta_1]\!] \triangle f^{-1}[\![\theta_1]\!]_{S'})$, so, by induction, included in $N_0 \cup N_1 \in \mathcal{N}$, for some N_0, N_1 in $\mathcal{N}$. Finally, for $\langle a\rangle_r\theta$:

$$\begin{array}{rcll} [\![\langle a\rangle_r\theta]\!] & := & \{s : h(s,\theta) \geq r\} & \\ & \sim_{\mathcal{N}} & \{s \mid h(s, f^{-1}[\![\theta]\!]_{S'}) \geq r\} & \\ & \sim_{\mathcal{N}} & f^{-1}\{s' \mid h'(s', [\![\theta]\!]_{S'}) \geq r\} & =: \ f^{-1}[\![\langle a\rangle_r\theta]\!]_{S'} \end{array}$$

where the second line uses Lemma 7.5 and the induction hypothesis, and the third line is because f is a coalgebra morphism. ∎

Note that Lemma 8.3 is also true for tests (see the proof of Lemma 8.1 for a definition of tests), indeed, using equations (1) and (3):

$$\begin{array}{rcl} f^{-1}(\cap_i[\![\theta_i]\!]_{S'}^{\epsilon_i}) \triangle \cap_i[\![\theta_i]\!]_S^{\epsilon_i} & = & (\cap_i f^{-1}([\![\theta_i]\!]_{S'})^{\epsilon_i}) \triangle (\cap_i[\![\theta_i]\!]_S^{\epsilon_i}) \\ & \subseteq & \cap_i(f^{-1}([\![\theta_i]\!]_{S'})^{\epsilon_i} \triangle [\![\theta_i]\!]_S^{\epsilon_i}) \\ & = & \cap_i(f^{-1}([\![\theta_i]\!]_{S'}) \triangle [\![\theta_i]\!]_S) \in \mathcal{N} \end{array}$$

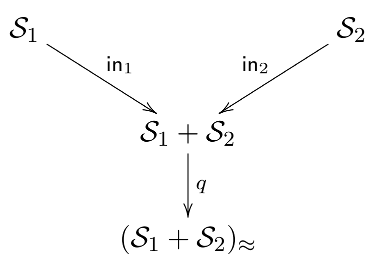
Corollary 8.4 *If $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a zigzag morphism, then $\mathcal{S} \approx_{\mathcal{L}} \mathcal{S}'$.*

Proof . Suppose $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a zigzag, we prove that f^{-1} restricts to an isomorphism between $\sigma([\![\mathcal{L}]\!]_1)/\mathcal{N}_1$ and $\sigma([\![\mathcal{L}]\!]_2)/\mathcal{N}_2$. First, since f is a zigzag, one has $f^{-1}[\![\theta]\!]_2 \sim_{\mathcal{N}_1} [\![\theta]\!]_1$ (by Lemma 8.3), so f^{-1} indeed defines an homomorphim from $\sigma([\![\mathcal{L}]\!]_2)/\mathcal{N}_2$ to $\sigma([\![\mathcal{L}]\!]_1)/\mathcal{N}_1$ (because $\sim$ is preserved under σ-algebraic operations and f^{-1}), and clearly f extends ρ. Moreover, since f is an epimorphism, f^{-1} is a monomorphism (by Prop. 6.9). To prove that f^{-1} is an epimorphism, it is enough to prove that

$$\mathcal{F} := \{A \in \sigma([\![\mathcal{L}]\!]_1) \mid \exists A' \in \sigma([\![\mathcal{L}]\!]_2) : A \sim_{\mathcal{N}} f^{-1}(A')\}$$

is a σ-algebra containing $[\![\mathcal{L}]\!]_1$. By Lemma 8.3, we do have $[\![\theta]\!] \in \mathcal{F}$. Clearly $\mathcal{F}$ is closed under complementation and contains the empty set; $\mathcal{F}$ is closed under countable intersection because if $A_i \sim_{\mathcal{N}} f^{-1}(A_i')$, then $\cap_i A_i \triangle \cap_i f^{-1}(A_i') \subseteq \cup_i(A_i \triangle f^{-1}(A_i')) \in \mathcal{N}$. ∎

We now prove the reverse direction. To do so, we plan to construct a cospan as follows. Given LMPns $\mathcal{S}_1 \approx_{\mathcal{L}} \mathcal{S}_2$,



where $\mathcal{S}_1 + \mathcal{S}_2$ is the disjoint sum LMPn (with the obvious definition) and $(\mathcal{S}_1 + \mathcal{S}_2)_\approx$ is the *logical quotient* of $\mathcal{S}_1 + \mathcal{S}_2$. In general, given an LMPn $\mathcal{S}$, its logical quotient is the LMPn $(S/_\approx, \Sigma_\approx, \mathcal{N}_\approx, h_\approx)$, $\approx$ being the logical equivalence of states, and writing q for the associated quotient map $q : S \to S/_\approx$:

$$\begin{array}{rcl} \Sigma_\approx & := & q(\sigma([\![\mathcal{L}]\!])) \\ \mathcal{N}_\approx & := & \{X \in \Sigma_\approx \mid q^{-1}(X) \in \mathcal{N}\} \\ h_\approx([s], X) & = & h(s, q^{-1}X) \end{array}$$

Proposition 8.5 *The logical quotient is an LMP and the quotient morphism is a zigzag morphism.*

Proof . We first prove that the quotient is an LMP. The fact that $h_{\approx}$ is well defined is routine, using the fact that measures that agree on a π-system (formulas) agree on the σ-algebra generated by it. We only have to check that $\Sigma_{\approx}$ and $\mathcal{N}_{\approx}$ are stable; the former is easy, and we prove only the latter. Let $X \in \mathcal{N}_{\approx}$; then $q^{-1}X \in \mathcal{N}$ by definition of $\mathcal{N}_{\approx}$, and hence $\{s \mid h_S(s, q^{-1}X) > 0\} \in \mathcal{N}$. Now we have

$$\begin{aligned} q^{-1}\{s' \in S_{\approx} \mid h_{\approx}(s', X) > 0\} &= \{s \mid h_{\approx}(q(s), X) > 0\} \\ &= \{s \mid h(s, q^{-1}X) > 0\} \in \mathcal{N}, \end{aligned}$$

and hence $\{s' \in S_{\approx} \mid h_{\approx}(s', X) > 0\} \in \mathcal{N}_{\approx}$.

The fact that q is a morphism of coalgebra is given by definition of $\Sigma_{\approx}$ and $h_{\approx}$ and by Lemma 7.4. That it is an epimorphism is given by definition of $\mathcal{N}_{\approx}$. ∎

Theorem 8.6 *$S_1 \approx_{\mathcal{L}} S_2$ if and only if S_1 and S_2 are event bisimilar.*

Proof . $\Leftarrow$: is given by Corollary 8.4.
$\Rightarrow$: we know that $q : S_1 + S_2 \to (S_1 + S_2)/_{\approx}$ is a zigzag. We have to prove that for $i = 1, 2$, $q \circ \mathsf{in}_i : S_i \to S_1 + S_2 \to (S_1 + S_2)/_{\approx}$ also are. We first prove that $q \circ \mathsf{in}_i$ are coalgebra morphisms. Let $X \in \Sigma_{\approx}$. Then for $s_i \in S_i$,

$$h_{\approx}(q \circ \mathsf{in}_i(s_i), X) = h_{1+2}(\mathsf{in}_i(s_i), q^{-1}X) = h_i(s_i, \mathsf{in}_i^{-1} q^{-1} X).$$

Finally, if $A \in \mathcal{N}_{\approx}$, we have $q^{-1}A \in \mathcal{N}_{1+2}$ which implies $\mathsf{in}_i^{-1}(q^{-1}A) \in \mathcal{N}_i$ for $i = 1, 2$.

It remains to prove that $q \circ \mathsf{in}_i$ is an epimorphism. Pick an X' in $\Sigma_{\approx}$ such that $(q \circ \mathsf{in}_i)^{-1}(X') \in \mathcal{N}_i$. We want to prove that X' is negligible, which by definition of $\mathcal{N}_{\approx}$, amounts to saying that $q^{-1}(X') \in \mathcal{N}_{1+2}$. Now this would follow clearly from $(q \circ \mathsf{in}_1)^{-1}(X') \in \mathcal{N}_1$ iff $(q \circ \mathsf{in}_2)^{-1}(X') \in \mathcal{N}_2$. Let us see why this last statement is true. By construction of the quotient, it downs to saying that for every $X \in \sigma([\![\mathcal{L}]\!]_{1+2})$, $\mathsf{in}_1^{-1}(X) \in \mathcal{N}_1$ iff $\mathsf{in}_2^{-1}(X) \in \mathcal{N}_2$.

Write $X_i := \mathsf{in}_i^{-1}(X)$ for X in $\sigma([\![\mathcal{L}]\!]_{1+2})$; since $\mathsf{in}_i^{-1}(\sigma([\![\mathcal{L}]\!]_{1+2})) = \sigma([\![\mathcal{L}]\!]_i)$, it follows that $X_i \in \sigma([\![\mathcal{L}]\!]_i)$. By hypothesis, there is an isomorphism $\hat{\rho}$ between $\sigma([\![\mathcal{L}]\!]_1)/\mathcal{N}_1$ and $\sigma([\![\mathcal{L}]\!]_2)/\mathcal{N}_2$ such that for all $\theta \in \mathcal{L}$, $\hat{\rho}([[\![\theta]\!]_1]) = [[\![\theta]\!]_2]$. Consider now the set:

$$\{X \mid \hat{\rho}([\mathsf{in}_1^{-1}(X)]) = [\mathsf{in}_2^{-1}(X)]\}$$

1) it is a σ-algebra, since $\hat{\rho}$ is a σ-complete morphism, 2) and since $\mathsf{in}_i^{-1}([\![\theta]\!]_{1+2}) = [\![\theta]\!]_i$, it contains $[\![\mathcal{L}]\!]_{1+2}$. Therefore $\hat{\rho}([X_1]) = [X_2]$ holds for all X in $\sigma([\![\mathcal{L}]\!]_{1+2})$, and in particular if $X_1 \in \mathcal{N}_1$, then $X_2 \in \mathcal{N}_2$.
∎

9 LMPs without states

It is clear that in the presence of the notion of negligible sets the role of states gets pushed to the background. In this section we investigate the possibility that LMPs are described without states.

As mentioned earlier, any lattice that corresponds to a σ-algebra over a set is a σ-complete Boolean algebra. We recall that, in this setting, a set of negligibles is a proper σ-ideal (of a σ-complete Boolean algebra), and, given any two morphisms $f, g : (S, \Sigma_S, \mathcal{N}_S) \to (T, \Sigma_T, \mathcal{N}_T)$ of $\mathbf{Mes}'$, and their corresponding homomorphisms of σ-complete Boolean algebra $\breve{f}^{-1}$, $\breve{g}^{-1} : \Sigma_T/\mathcal{N}_T \to \Sigma_S/\mathcal{N}_S$, we have:

$$f \sim g \quad \Leftrightarrow \quad \breve{f}^{-1} = \breve{g}^{-1}$$

This fact together with our *event* bisimulation approach, enables us to capture our notions of LMPns and bisimulation in a purely algebraic framework. An LMPn will be defined only by the quotient of its σ-algebra over its σ-ideal of negligibles (viewed as an abstract σ-complete Boolean algebra quotient) and by some "kernels" related to our logic $\mathcal{L}$: no state will be involved in the definition.

9.1 Boolean algebras and σ-complete Boolean algebras

A *σ-Boolean algebra* $(\Sigma, 0, 1, \vee_\omega, \wedge_\omega, \neg)$ is a Boolean algebra closed under countable products (infima) and sums (suprema).

Let $\sigma\mathbf{BA}$ denote the category of σ-complete Boolean algebras equipped with σ-complete homomorphisms of Boolean algebras, i.e., arrows ρ such that:

$$\begin{aligned}
&\rho(0) = 0,\ \rho(1) = 1 \\
&\rho(\wedge_{i\in\omega} a_i) = \wedge_{i\in\omega}\rho(a_i) \\
&\rho(\vee_{i\in\omega} a_i) = \vee_{i\in\omega}\rho(a_i) \\
&\rho(\neg a) = \neg\rho(a)
\end{aligned}$$

so commuting to countable infima and complement (therefore to countable suprema).

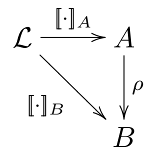
A Boolean algebra B is called an *algebra of sets* (resp. a *σ-algebra of sets*) if $B \subseteq \mathcal{P}(X)$ for some X, and B is closed under complementation and under finite (resp. countable) intersections. A σ-algebra of sets is therefore exactly a σ-algebra.

By the famous Stone-Duality theorem [Joh82, Mon89], we know that every Boolean algebra is isomorphic to some algebra of sets. Unfortunately this is no longer true for σ-complete algebras; there are σ-complete algebras that are not (σ-complete) isomorphic to any σ-algebra. However, any σ-complete algebra is (σ-complete) isomorphic to the quotient of a σ-algebra and a σ-ideal, see the Loomis-Sikorski theorem [Mon89]. From that perspective, it seems therefore natural to generalise our notion of LMPns to a purely algebraic setting.

9.2 Abstract LMPns

We want the underlying space of an abstract LMPn to be some σ-complete Boolean algebra A. We also need something that will take the place of the transition kernels. In fact, we know from the previous section that, since the logic characterizes bisimulation, it is a good candidate to help us abstract LMPn while encoding all the information in the transition kernels. For every logical formula of $\mathcal{L}$, there is a corresponding set of A. We define the category **LMPa** of abstract LMPs with negligibles.

Definition 9.1 *An* ***object*** *of* ***LMPa*** *is a function $[\![\cdot]\!]$ from $\mathcal{L}$ to a σ-complete Boolean algebra A. A* ***morphism*** *from $\mathcal{L} \to A$ to $\mathcal{L} \to B$ is an homomorphism $\rho : A \to B$ of σ-complete Boolean* *algebras that makes the following diagram commute.*



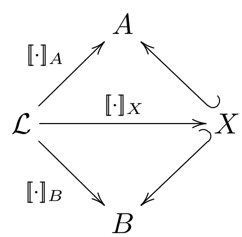
We will define a functor from the category **LMPn** to **LMPa**. The category **LMPn** has LMPn as objects and coalgebra morphisms as morphisms.

Definition 9.2 *The contravariant functor* $\mathcal{F} : \mathbf{LMPn} \to \mathbf{LMPa}$ *is defined as follows:*

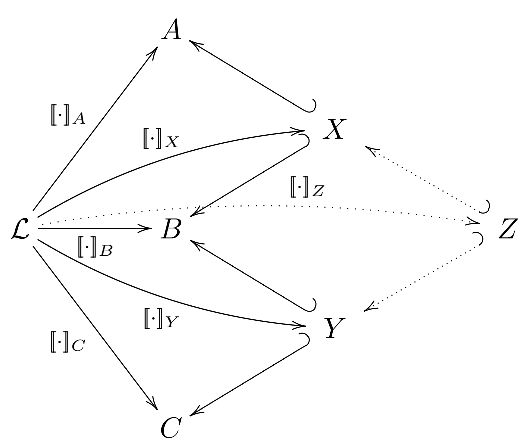
- $\mathcal{F}(S, \Sigma, \mathcal{N}, h) = [\![\cdot]\!]_{\Sigma/\mathcal{N}} : \mathcal{L} \to \Sigma/\mathcal{N}$ *where* $[\![\phi]\!]_{\Sigma/\mathcal{N}}$ *is the equivalence class of* $[\![\phi]\!]_{\mathcal{S}}$.
- *for* $f : \mathcal{S} \to \mathcal{T}$, $\mathcal{F}(f) = \breve{f}^{-1} : \Sigma_T/\mathcal{N}_T \to \Sigma_S/\mathcal{N}_S$

Proof . It is easy to prove that $\mathcal{F}(\mathsf{id}_S) = \mathsf{id}_{\Sigma_\mathcal{S}/\mathcal{N}_S}$ and $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$. It remains to prove that $\mathcal{F}(f)$ makes the triangle commute. Let $\phi \in \mathcal{L}$. We want to show that $F(f) \circ [\![\phi]\!]_{\Sigma_T/\mathcal{N}_T} = [\![\phi]\!]_{\Sigma_S/\mathcal{N}_S}$. But the left part is simply $\breve{f}^{-1}([\![\phi]\!]_{\Sigma_T/\mathcal{N}_T})$ and by Lemma 8.3, we have that $f^{-1}([\![\phi]\!]_T) \triangle [\![\phi]\!]_S \in \mathcal{N}$ which implies the result. ∎

Definition 9.3 *Two abstract LMPs are bisimilar if they are related by a span of monomorphisms. In other words, a bisimulation between A and B is represented in the following commutative diagram.*



Since pullbacks exist in **LMPa** (being inherited from σ**BA**) and since they preserve monomorphisms, bisimulation is an equivalence. The proof is summarized in the following commutative diagram.



where $Z = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and $[\![\cdot]\!]_Z ::= ([\![\cdot]\!]_X, [\![\cdot]\!]_Y)$ are given by a pullback construction.

The functor $\mathcal{F}$ acts on zigzag morphisms as we would expect, it yields a monomorphism of Boolean σ-algebra.

Proposition 9.4 *Let $f : \mathcal{S} \to \mathcal{T}$ be a zigzag, then $\mathcal{F}(f)$ is a monomorphism of coalgebras.*

Proof . We have to show that $\ker(\mathcal{F}(f)) = 0_{\Sigma_T/\mathcal{N}_T}$. A zigzag morphism being an epimorphism of coalgebra, we have, from Proposition 6.9, that $\forall X \in \Sigma_T : f^{-1}(X) \in \mathcal{N}_S \Rightarrow X \in \mathcal{N}_T$, which yields the result. ∎

This proposition shows that abstract LMPn objects are a generalization of ordinary LMPn objects.

Corollary 9.5 *If two LMPns $\mathcal{S}$ and $\mathcal{T}$ are bisimilar, then $\mathcal{F}(\mathcal{S})$ and $\mathcal{F}(\mathcal{T})$ also are.*

Proof . The result follows from the fact that a cospan of zigzag morphisms between $\mathcal{S}$ and $\mathcal{T}$ induces a span of monomorphisms between $\mathcal{F}(\mathcal{S})$ and $\mathcal{F}(\mathcal{T})$. ∎

The simplicity of the construction and the proofs of this section is not serendipitous. In fact, for a few of the results of the previous sections, among all the possible definitions that we could have chosen, the exact one that permitted us to prove the results were given naturally from this idea that LMPn should be viewed as abstract LMPs. In particular, the definition of equivalence between morphisms of $\mathbf{Mes}'$ (see Eq. 8) and Definition 8.2 of logical equivalence.

It is important to note that these ideas could be extended to other equivalences, by replacing $\mathcal{L}$ by other suitable logics. Moreover, there is also a possibility (especially due to the Loomis-Sikorski theorem) that this idea of abstract LMPns gives rise to a full theory of pointless LMPn, and would form a Stone-type duality with the ordinary LMPs in the same way that Frames (or Locales), are "pointless" topological spaces [Joh82].

10 Conclusion

The enterprise described in the present paper consists of three parts. In the first part we develop the idea that one should use cospans rather than spans or relations in formulating probabilistic bisimulation. Others [**?**] have also voiced similar thoughts. The most important mathematical benefit of this shift is that the logical characterization of bisimulation can be established for LMPs in a perfectly general way, without, for example, any assumptions about the state space being analytic. We take an explicitly coalgebraic view of LMPs in order to formulate this notion of bisimulation.

Given the categorical machinery at hand we use it to refine the notion of bisimulation so as to allow one to talk about systems being bisimilar "almost everywhere". Typically the phrase "almost everywhere" or "almost surely" are used with reference to some ambient measure. However, we work with an axiomatic notion of "ideal of negligible sets". We use this to modify the monad and the coalgebras so as to accomodate the new notion of "almost sure bisimilarity." So far we have essentially reviewed our previous work [DDLP05]. However at this point we make a significant departure. We define a new notion of equivalence of arrows. This allows one to obtain a very pleasing connection with bisimulation and logic.

Perhaps the most suggestive idea to come out of this work is the idea that states are not as central as hitherto imagined. Rather one should work with abstract notions of LMPs without explicit mention of the state space. This is analogous to the idea of pointless topology so forcefully argued by Johnstone [Joh82] where one works with the lattice of opens viewed as a frame. Thus there is a Stone-type duality at work here which we have just begun to understand. Kozen [Koz85] has already discussed such a Stone type duality for probabilistic programs. In his case the duality is between a probabilistic state transition system (viewed as giving the forward semantics of a programming language) and a probabilistic analogue of a predicate transformer semantics, which gives a "backward" flowing semantics. The connection between these two is exactly a Stone-type duality. There are many such duality notions in the air at the moment in a variety of areas.

These duality ideas have proven useful already. Work on metrics for LMPs [DGJP99, **?**] was inspired by Kozen's work. A very important possible payoff for ideas of abstract or pointless LMPs is a more satisfactory approximation theory. Danos and Plotkin have already investigated such ideas in unpublished work.

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